COSMIC RAY ACCELERATION

LECTURE 1: BASICS

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OUTLINE OF THE MINI-COURSE

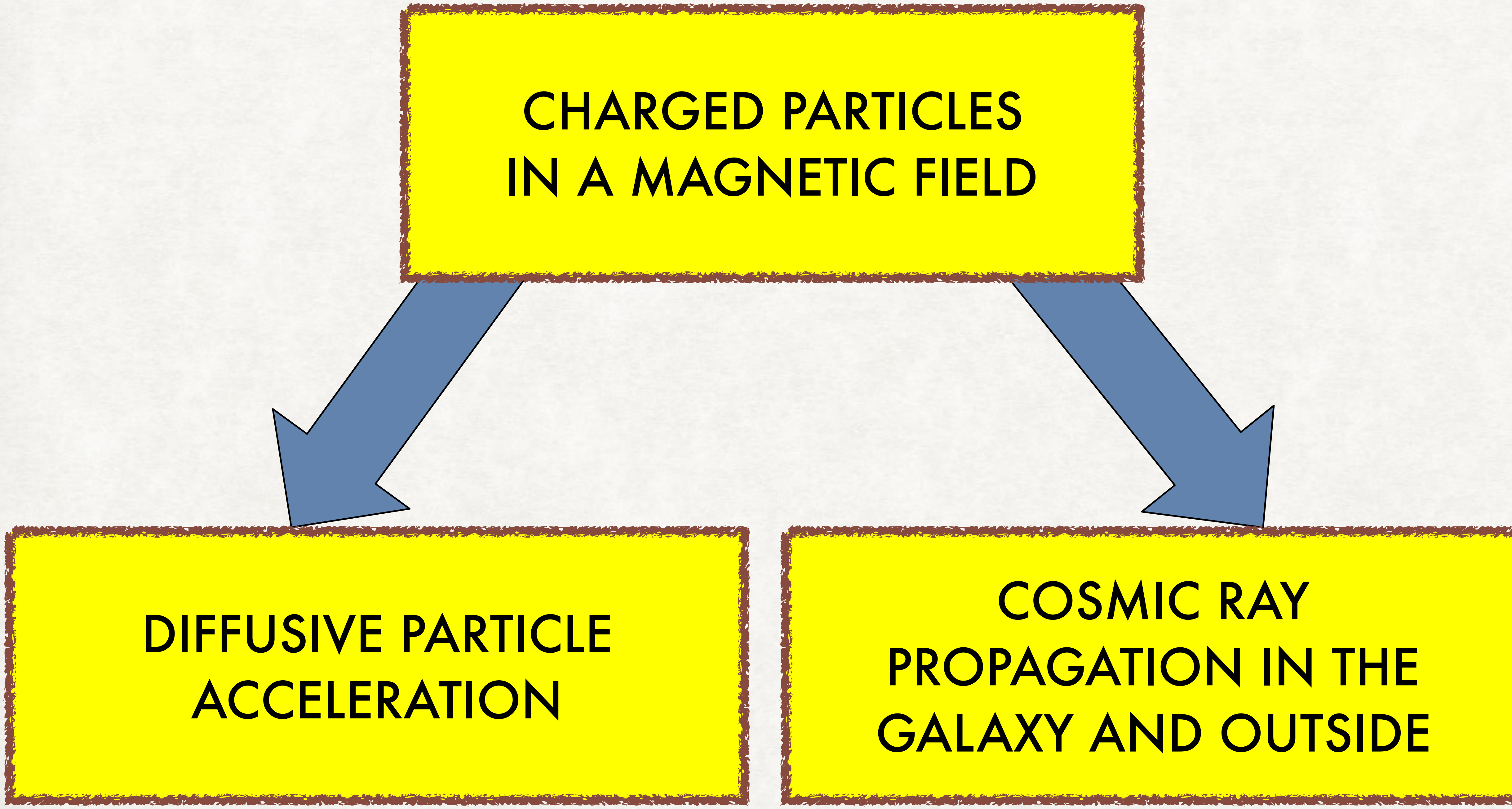
- **Basics of particle transport in the presence of magnetic perturbations**
- **Second order Fermi acceleration**
- **Shock waves in astrophysics**

- **First order Fermi acceleration - test particle theory**
  - *Spectrum from statistical approach*
  - *Spectrum from the transport equation*
  - *Maximum energy of accelerated particles*

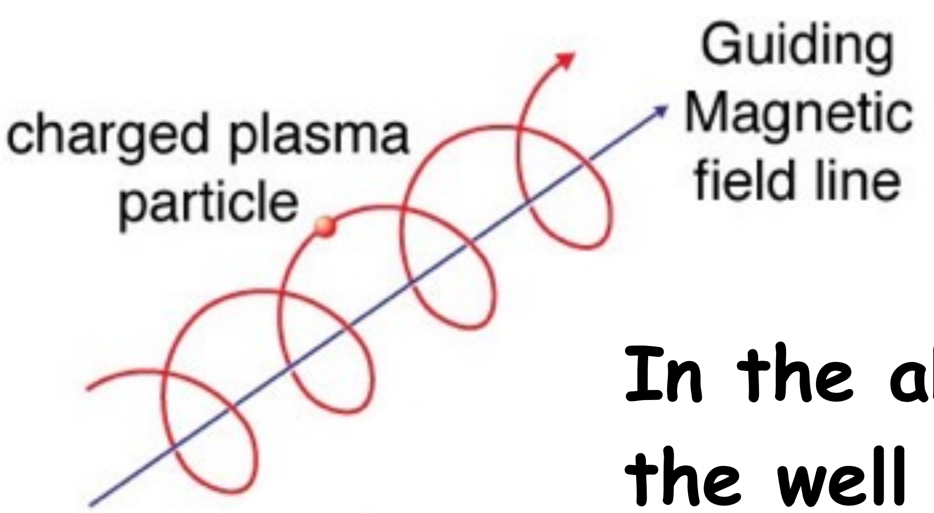
- **First order Fermi acceleration - basic non-linear theory**
  - *non linear dynamical reaction*
  - *cosmic ray induced plasma instabilities -* E_{max}
  - *DSA in the presence of neutral hydrogen*

- **Basics of acceleration to ultra high energies**

COSMIC RAY TRANSPORT



CHARGED PARTICLES IN A REGULAR B FIELD



$$\frac{d\vec{p}}{dt} = q\left[\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right]$$

In the absence of an electric field one obtains the well known solution:

$$p_z = \text{Constant}$$

$$v_x = V_0 \cos[\Omega\, t]$$

$$v_y = V_0 \sin[\Omega\, t]$$

LARMOR FREQUENCY

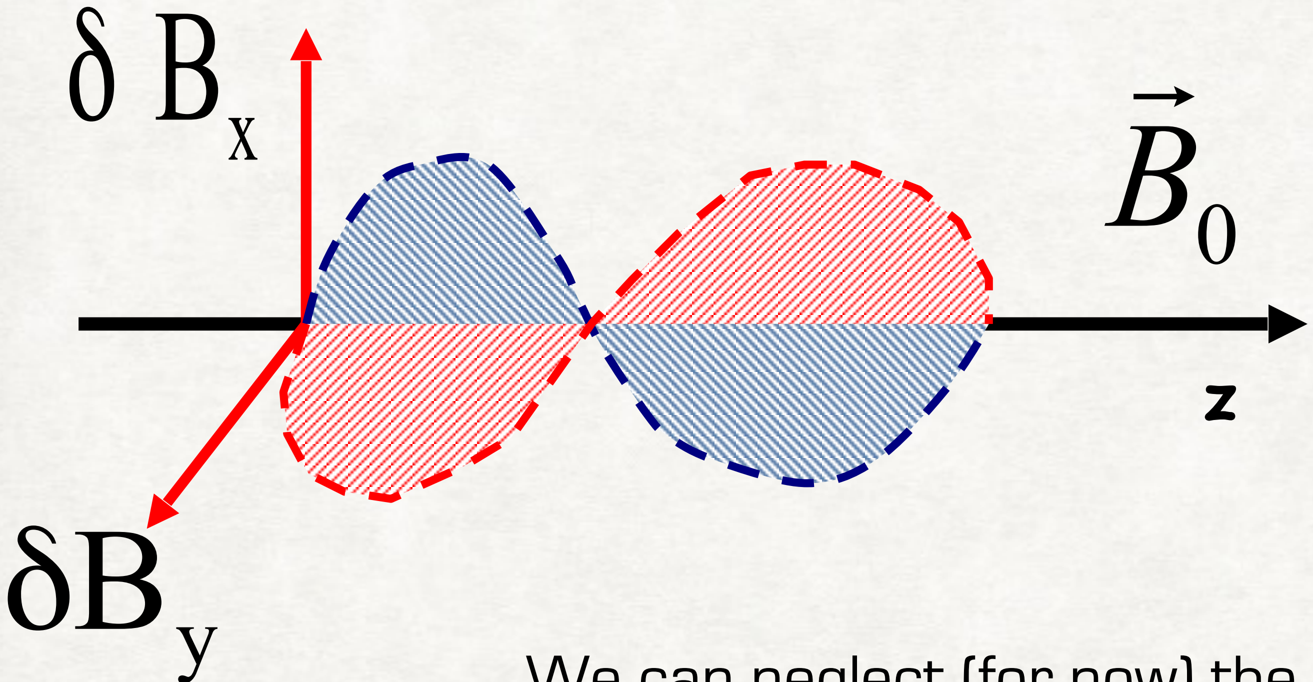
$$\Omega = \frac{q\, B_0}{m\, c\, \gamma}$$

A FEW THINGS TO KEEP IN MIND

- THE MAGNETIC FIELD DOES NOT CHANGE PARTICLE ENERGY -> NO ACCELERATION BY B FIELDS

- A RELATIVISTIC PARTICLE MOVES IN THE Z DIRECTION ON AVERAGE AT C/3

MOTION OF A PARTICLE IN A WAVY FIELD



Let us consider an Alfven wave propagating in the z direction:

$$\delta B << B_0 \qquad \delta\vec{B} \perp \vec{B}_0$$

We can neglect (for now) the electric field associated with the wave, or in other words we can sit in the reference frame of the wave:

$$\frac{d\vec{p}}{dt} = q\frac{\vec{v}}{c} \times (\vec{B}_0 + \delta\vec{B})$$

THIS CHANGES ONLY THE X AND Y COMPONENTS OF THE MOMENTUM

THIS TERM CHANGES ONLY THE DIRECTION OF $P_Z = P\mu$

Remember that the wave typically moves with the Alfven speed:

$$v_a = \frac{B}{(4\pi\rho)^{1/2}} = 2 \times 10^6 B_\mu n_1^{-1/2} \ cm/s$$

Alfven waves have frequencies << ion gyration frequency $\Omega_p = qB/m_p c$

It is therefore clear that for a relativistic particle these waves, in first approximation, look like static waves.

The equation of motion can be written as:

$$\frac{d\vec{p}}{dt} = \frac{q}{c}\vec{v} \times (\vec{B}_0 + \vec{\delta}B)$$

If to split the momentum in parallel and perpendicular, the perpendicular component cannot change in modulus, while the parallel momentum is described by

$$\frac{dp \parallel}{dt} = \frac{q}{c}|\vec{v}_\perp \times \delta\vec{B}| \qquad p_{||} = p \ \mu$$

$$\frac{d\mu}{dt} = \frac{q}{pc} v(1-\mu^2)^{1/2} \delta B cos(\Omega t - kx + \psi)$$

Wave form of the magnetic field with a random phase and frequency

$$\Omega = qB_0/mc\gamma$$

Larmor frequency

In the frame in which the wave is at rest we can write

$$x = v\mu t$$

$$\frac{d\mu}{dt} = \frac{q}{pc} v(1-\mu^2)^{1/2} \delta B cos\left[(\Omega - kv\mu)t + \psi\right]$$

It is clear that the mean value of the pitch angle variation over a long enough time vanishes

$$\langle \Delta\mu \rangle_t = 0$$

We want to see now what happens to

$$\langle \Delta\mu \Delta\mu \rangle$$

Let us first average upon the random phase of the waves:

$$\langle \Delta\mu(t')\Delta\mu(t")\rangle_\psi = \frac{q^2v^2(1-\mu^2)\delta B^2}{2c^2p^2} cos\left[(\Omega - kv\mu)(t' - t")\right]$$

And integrating over time:

$$\langle \Delta\mu\Delta\mu\rangle_t = \frac{q^2v^2(1-\mu^2)\delta B^2}{2c^2p^2}\int dt' \int dt" cos\left[(\Omega - kv\mu)(t' - t")\right]$$

$$= \frac{q^2v(1-\mu^2)\delta B^2}{c^2p^2\mu}\delta(k - \Omega/v\mu)\Delta t$$

↑

RESONANCE

Many waves

IN GENERAL ONE DOES NOT HAVE A SINGLE WAVE BUT RATHER A POWER SPECTRUM:

$$P(k) = B_k^2/4\pi$$

THEREFORE INTEGRATING OVER ALL OF THEM:

$$\langle \frac{\Delta\mu\Delta\mu}{\Delta t} \rangle = \frac{q^2(1-\mu^2)\pi}{m^2c^2\gamma^2} \frac{1}{v\mu} 4\pi \int dk \frac{\delta B(k)^2}{4\pi} \delta(k - \Omega/v\mu)$$

OR IN A MORE IMMEDIATE FORMALISM:

$$\left\langle \frac{\Delta\mu\Delta\mu}{\Delta t} \right\rangle = \frac{\pi}{2}\Omega\,(1-\mu^2)k_{res}F(k_{res})$$

$$k_{res} = \frac{\Omega}{v\mu}$$

RESONANCE!!!

DIFFUSION COEFFICIENT

THE RANDOM CHANGE OF THE PITCH ANGLE IS DESCRIBED BY A DIFFUSION COEFFICIENT

$$D_{\mu\mu} = \left\langle \frac{\Delta\theta\Delta\theta}{\Delta t} \right\rangle = \frac{\pi}{4} \Omega k_{res} F(k_{res})$$

FRACTIONAL POWER $(\delta B/B_0)^2$ $=G(k_{res})$

THE DEFLECTION ANGLE CHANGES BY ORDER UNITY IN A TIME:

$$\tau \approx \frac{1}{\Omega\, G(k_{res})}$$

⟹

PATHLENGTH FOR DIFFUSION ~ $v\tau$

$$\left\langle \frac{\Delta z \Delta z}{\Delta t} \right\rangle \approx v^2\, \tau = \frac{v^2}{\Omega\, G(k_{res})}$$

SPATIAL DIFFUSION COEFF.

PARTICLE SCATTERING

- EACH TIME THAT A RESONANCE OCCURS THE PARTICLE CHANGES PITCH ANGLE BY $\Delta\theta \sim \delta B/B$ WITH A RANDOM SIGN

- THE RESONANCE OCCURS ONLY FOR RIGHT HAND POLARIZED WAVES IF THE PARTICLES MOVES TO THE RIGHT (AND VICEVERSA)

- THE RESONANCE CONDITION TELLS US THAT 1) IF $k \ll 1/r_L$ PARTICLES SURF ADIABATICALLY AND 2) IF $k \gg 1/r_L$ PARTICLES HARDLY FEEL THE WAVES

What Equations for Diffusion?

BASIC FORMALISM

$$f(\vec{p}, \vec{x}, t)$$

DISTRIBUTION FUNCTION OF PARTICLES WITH MOMENTUM P AT THE POSITION X AT TIME T

$$\Psi(\vec{p}, \Delta\vec{p})$$

PROBABILITY THAT A PARTICLE WITH MOMENTUM P CHANGES ITS MOMENTUM BY DELTA P

$$\int d\Delta\vec{p}\, \Psi(\vec{p}, \Delta\vec{p}) = 1$$

In general we can write:

$$f(\vec{p}, \vec{x} + \vec{v}\Delta t, t + \Delta t) = \int d\Delta\vec{p}\; f(\vec{p} - \Delta\vec{p}, \vec{x}, t)\Psi(\vec{p} - \Delta\vec{p}, \Delta\vec{p})$$

In the limit of small momentum changes we can Taylor – expand:

$$f(\vec{p}, \vec{x} + \vec{v}\Delta t, t + \Delta t) = f(\vec{p}, \vec{x}, t) + \left(\vec{v}\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t}\right)\Delta t$$

$$f(\vec{p} - \Delta\vec{p}, \vec{x}, t) = f(\vec{p}, \vec{x}, t) - \frac{\partial f}{\partial \vec{p}}\Delta\vec{p} + \frac{1}{2}\Delta\vec{p}\Delta\vec{p}\frac{\partial^2 f}{\partial \vec{p}^2}$$

$$\Psi(\vec{p} - \Delta\vec{p}, \Delta\vec{p}) = \Psi(\vec{p}, \Delta\vec{p}) - \frac{\partial \Psi}{\partial \vec{p}}\Delta\vec{p} + \frac{1}{2}\Delta\vec{p}\Delta\vec{p}\frac{\partial^2 \Psi}{\partial \vec{p}^2}$$

Substituting in the first Equation:

$$f + \Delta t\left(\vec{v}\frac{\partial f}{\partial \vec{x}} + \frac{\partial f}{\partial t}\right) = \int d\Delta\vec{p}\left(f(\vec{p},\vec{x},t) - \frac{\partial f}{\partial \vec{p}}\Delta\vec{p} + \frac{1}{2}\Delta\vec{p}\Delta\vec{p}\frac{\partial^2 f}{\partial \vec{p}^2}\right)\left(\Psi(\vec{p},\vec{x},t) - \frac{\partial \Psi}{\partial \vec{p}}\Delta\vec{p} + \frac{1}{2}\Delta\vec{p}\Delta\vec{p}\frac{\partial^2 \Psi}{\partial \vec{p}^2}\right)$$

Recall that $\int d\Delta\vec{p}\;\Psi(\vec{p},\Delta\vec{p}) = 1$

$$\frac{\partial f}{\partial t} + \vec{v}\frac{\partial f}{\partial \vec{x}} = -\frac{\partial}{\partial \vec{p}}\left[f\langle\frac{\Delta\vec{p}}{\Delta t}\rangle\right] + \frac{1}{2}\frac{\partial}{\partial \vec{p}}\left[\frac{\partial}{\partial \vec{p}}\left(\langle\frac{\Delta\vec{p}\Delta\vec{p}}{\Delta t}\rangle f\right)\right]$$

$$\langle\frac{\Delta\vec{p}}{\Delta t}\rangle = \frac{1}{\Delta t}\int d\Delta\vec{p}\;\Delta\vec{p}\;\Psi(\vec{p},\Delta\vec{p})$$

$$\langle\frac{\Delta\vec{p}\Delta\vec{p}}{\Delta t}\rangle = \frac{1}{\Delta t}\int d\Delta\vec{p}\;\Delta\vec{p}\;\Delta\vec{p}\;\Psi(\vec{p},\Delta\vec{p})$$

We can now use a sort of Principle of Detailed Balance:

$$\Psi(\vec{p}, -\Delta\vec{p}) = \Psi(\vec{p} - \Delta\vec{p}, \Delta\vec{p})$$

and expanding the RHS:

$$\Psi(\vec{p}, -\Delta\vec{p}) = \Psi(\vec{p}, \Delta\vec{p}) - \Delta\vec{p}\frac{\partial\Psi}{\partial\vec{p}} + \frac{1}{2}\Delta\vec{p}\Delta\vec{p}\frac{\partial^2\Psi}{\partial\vec{p}^2}$$

And integrating in Delta p:

$$1 = 1 - \frac{\partial}{\partial\vec{p}}\langle\frac{\Delta\vec{p}}{\Delta t}\rangle + \frac{1}{2}\frac{\partial}{\partial\vec{p}}\frac{\partial}{\partial\vec{p}}\langle\frac{\Delta\vec{p}\Delta\vec{p}}{\Delta t}\rangle$$

$$\Downarrow$$

$$\langle\frac{\Delta\vec{p}}{\Delta t}\rangle - \frac{1}{2}\frac{\partial}{\partial\vec{p}}\langle\frac{\Delta\vec{p}\Delta\vec{p}}{\Delta t}\rangle = \text{Constant}$$

We shall see later that the terms in this Eq. vanish for p→0, therefore the Constant must be zero and we have:

$$\langle \frac{\Delta \vec{p}}{\Delta t} \rangle = \frac{1}{2} \frac{\partial}{\partial \vec{p}} \langle \frac{\Delta \vec{p} \Delta \vec{p}}{\Delta t} \rangle$$

$$\frac{\partial f}{\partial t} + \vec{v} \frac{\partial f}{\partial \vec{x}} = -\frac{\partial}{\partial \vec{p}} \left[f \langle \frac{\Delta \vec{p}}{\Delta t} \rangle \right] + \frac{1}{2} \frac{\partial}{\partial \vec{p}} \left[\frac{\partial}{\partial \vec{p}} \left(\langle \frac{\Delta \vec{p} \Delta \vec{p}}{\Delta t} \rangle f \right) \right]$$

↓

$$\frac{\partial f}{\partial t} + \vec{v} \frac{\partial f}{\partial \vec{x}} = \frac{\partial}{\partial \vec{p}} \left[D_{pp} \frac{\partial f}{\partial \vec{p}} \right] \qquad D_{pp} = \frac{1}{2} \langle \frac{\Delta \vec{p} \Delta \vec{p}}{\Delta t} \rangle$$

BOLTZMANN EQUATION

COLLISION TERM

IN ONE SPATIAL DIMENSION ONE EASILY OBTAINS:

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu}\left[D_{\mu\mu}\frac{\partial f}{\partial \mu}\right]$$

WHERE

$$D_{\mu\mu} = \frac{1}{2}\langle\frac{\Delta\mu\Delta\mu}{\delta t}\rangle$$

IS THE PITCH ANGLE DIFFUSION COEFFICIENT.

THE PREVIOUS EQUATION CAN BE VIEWED AS THE BOLTZMANN EQUATION WITH A SCATTERING TERM DEFINED BY DIFFUSION.

From pitch to Spatial Diffusion

IT IS INTUITIVELY CLEAR HOW A PARTICLE THAT IS DIFFUSING IN ITS PITCH ANGLE MUST BE ALSO DIFFUSING IN SPACE. LET US SEE HOW THE TWO ARE RELATED TO EACH OTHER BY INTEGRATING THE BOLTZMANN EQUATION IN PITCH ANGLE:

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu}\left[D_{\mu\mu}\frac{\partial f}{\partial \mu}\right]$$

$$f_0(p,t,z) = \frac{1}{2}\int_{-1}^{1} d\mu f(p,t,\mu,z)$$

ISOTROPIC PART OF THE PARTICLE DISTRIBUTION FUNCTION. FOR MOST PROBLEMS THIS IS ALSO VERY CLOSE TO THE ACTUAL DISTRIBUTION FUNCTION

$$\frac{\partial f_0}{\partial t} + \frac{1}{2}v\int_{-1}^{1} d\mu\mu \frac{\partial f}{\partial z} \equiv 0$$

ONE CAN SEE THAT THE QUANTITY

$$J = \frac{1}{2} v \int_{-1}^{1} d\mu \mu f$$

BEHAVES AS A PARTICLE CURRENT, AND THE BOLTMANN EQUATION BECOMES:

$$\frac{\partial f_0}{\partial t} = -\frac{\partial J}{\partial z}$$

NOTICE THAT YOU CAN ALWAYS WRITE:

$$\mu = -\frac{1}{2} \frac{\partial}{\partial \mu} \left(1 - \mu^2\right)$$

WITH THIS TRICK:

$$J = \frac{1}{2} v \int_{-1}^{1} d\mu \mu f = \frac{v}{4} \int_{-1}^{1} d\mu (1-\mu^2) \frac{\partial f}{\partial \mu}$$

RECONSIDER THE INITIAL EQUATION

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu} \left[D_{\mu\mu} \frac{\partial f}{\partial \mu} \right]$$

AND INTEGRATE IT AGAIN FROM -1 TO m:

$$\frac{\partial}{\partial t} \int_{-1}^{\mu} d\mu' f + \int_{-1}^{\mu} d\mu' v \mu' \frac{\partial f}{\partial z} = D_{\mu\mu} \frac{\partial f}{\partial \mu}$$

AND MULTIPLYING BY $(1-\mu^2)/D_{\mu\mu}$

$$(1-\mu^2)\frac{\partial f}{\partial \mu} = \frac{1-\mu^2}{D_{\mu\mu}}\frac{\partial}{\partial t}\int_{-1}^{\mu} d\mu' f + \frac{1-\mu^2}{D_{\mu\mu}}\int_{-1}^{\mu} d\mu' v\mu' \frac{\partial f}{\partial z}$$

NOW RECALL THAT THE DISTRIBUTION FUNCTION TENDS TO ISOTROPY, SO THAT AT THE LOWEST ORDER IN THE ANISOTROPY ONE HAS:

$$(1-\mu^2)\frac{\partial f}{\partial \mu} = \frac{1-\mu^2}{D_{\mu\mu}}\frac{\partial f_0}{\partial t}(1+\mu) + \frac{1-\mu^2}{D_{\mu\mu}}\frac{1}{2}v(\mu^2-1)\frac{\partial f_0}{\partial z}$$

AND RECALLING THE DEFINITION OF CURRENT:

$$J = \frac{v}{4}\frac{\partial f_0}{\partial t}\int_{-1}^{1} d\mu \frac{1-\mu^2}{D_{\mu\mu}}(1+\mu) - \frac{v^2}{8}\frac{\partial f_0}{\partial z}\int_{-1}^{1} d\mu \frac{(1-\mu^2)^2}{D_{\mu\mu}} = \kappa_t \frac{\partial f_0}{\partial t} - \kappa_z \frac{\partial f_0}{\partial z}$$

USING THE TRANSPORT EQ IN TERMS OF CURRENT:

$$J = -\kappa_t \frac{\partial J}{\partial z} - \kappa_z \frac{\partial f_0}{\partial z}$$

NOW WE RECALL THE TRANSPORT EQUATION IN CONSERVATIVE FORM:

$$\frac{\partial f_0}{\partial t} = -\frac{\partial J}{\partial z}$$

AND PUTTING THINGS TOGETHER:

$$\frac{\partial f_0}{\partial t} = -\frac{\partial}{\partial z}\left[-\kappa_t \frac{\partial J}{\partial z} - \kappa_z \frac{\partial f_0}{\partial z}\right]$$

BUT IT IS EASY TO SHOW THAT THE FIRST TERM MUST BE NEGLIGIBLE:

$$J = \frac{v}{2}\int_{-1}^{1} d\mu \mu f_0(1+\delta\mu) = \frac{1}{3} v \delta f_0 \ll v f_0 \qquad \delta \ll 1$$

IT FOLLOWS THAT THE ISOTROPIC PART OF THE DISTRIBUTION FUNCTION MUST SATISFY THE DIFFUSION EQUATION:

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial z}\left[\kappa_z \frac{\partial f_0}{\partial z}\right]$$

DIFFUSION EQUATION

$$\kappa_z = \frac{v^2}{8}\int_{-1}^{1} d\mu \frac{(1-\mu^2)^2}{D_{\mu\mu}} = \frac{1}{3} v \lambda_{\parallel}$$

SPATIAL DIFFUSION COEFFICIENT

A MORE GENERAL RESULT

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\left[D\frac{\partial f}{\partial x}\right] - u\frac{\partial f}{\partial x} + \frac{1}{3}\frac{du}{dx}\,p\frac{\partial f}{\partial p} + Q(x,p,t)$$

Time Dep. **Diffusion** **Advection** **compression** **Injection**

THIS EQUATION, THOUGH IN ONE DIMENSION, CONTAINS ALL THE MAIN EFFECTS DESCRIBED BY MORE COMPLEX TREATMENTS

1. ***TIME DEPENDENCE***
2. ***DIFFUSION (EVEN SPACE AND MOMENTUM DEPENDENCE)***
3. ***ADVECTION (EVEN WITH A SPACE DEPENDENT VELOCITY)***
4. ***COMPRESSION AND DECOMPRESSION***
5. ***INJECTION***

IT DOES NOT INCLUDE 2nd ORDER AND SPALLATION, BUT EASY TO INCLUDE

IT APPLIES EQUALLY WELL TO TRANSPORT OF CR IN THE GALAXY OR TO CR ACCELERATION AT A SUPERNOVA SHOCK

ACCELERATION OF NONTHERMAL PARTICLES

The presence of non-thermal particles is ubiquitous in the Universe (solar wind, Active galaxies, supernova remnants, gamma ray bursts, Pulsars, micro-quasars)

WHEREVER THERE ARE MAGNETIZED PLASMAS THERE ARE NON-THERMAL PARTICLES

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PARTICLE ACCELERATION

BUT THERMAL PARTICLES ARE USUALLY DOMINANT, SO WHAT DETERMINES THE DISCRIMINATION BETWEEN THERMAL AND ACCELERATED PARTICLES?

INJECTION

ALL ACCELERATION MECHANISMS ARE ELECTROMAGNETIC IN NATURE

MAGNETIC FIELD CANNOT MAKE WORK ON CHARGED PARTICLES THEREFORE ELECTRIC FIELDS ARE NEEDED FOR ACCELERATION TO OCCUR

REGULAR ACCELERATION

THE ELECTRIC FIELD IS LARGE SCALE:

$$\langle \vec{E} \rangle \neq 0$$

STOCHASTIC ACCELERATION

THE ELECTRIC FIELD IS SMALL SCALE:

$$\langle \vec{E} \rangle = 0 \quad \langle \vec{E}^2 \rangle \neq 0$$

REGULAR ACCELERATION

$$\langle \vec{E} \rangle \neq 0$$

Very special conditions are necessary in Astrophysical environments in order to achieve this condition, because of the high electrical conductivity of astrophysical plasmas.
Few exceptions:

UNIPOLAR INDUCTOR: this occurs in the case of rotating magnetic fields, such as in pulsars, rotating black holes. An electric potential is established between the surface of the rotating object (neutrons star, BH) and infinity. The potential difference is usable only in places (gaps) where the condition $\vec{E} \cdot \vec{B} = 0$ is violated. MHD is broken in the gaps.

RECONNECTION: Locally, regions with opposite orientation of magnetic field merge, giving rise to a net local electric field E~LB, where L is the size of the reconnection region. It occurs in the sun and solar wind, but probably also in the magnetosphere of rotating neutron stars and BHs.

STOCHASTIC ACCELERATION

$$\langle \vec{E} \rangle = 0 \quad \langle \vec{E}^2 \rangle \neq 0$$

Most acceleration mechanisms that are operational in astrophysical environments are of this type. We have seen that the action of random magnetic fluctuations is that of scattering particles when the resonance is achieved. In other words, the particle distribution is isotropized in the reference frame of the wave.

Although in the reference frame of the waves the momentum is conserved (B does not make work) in the lab frame the particle momentum changes by

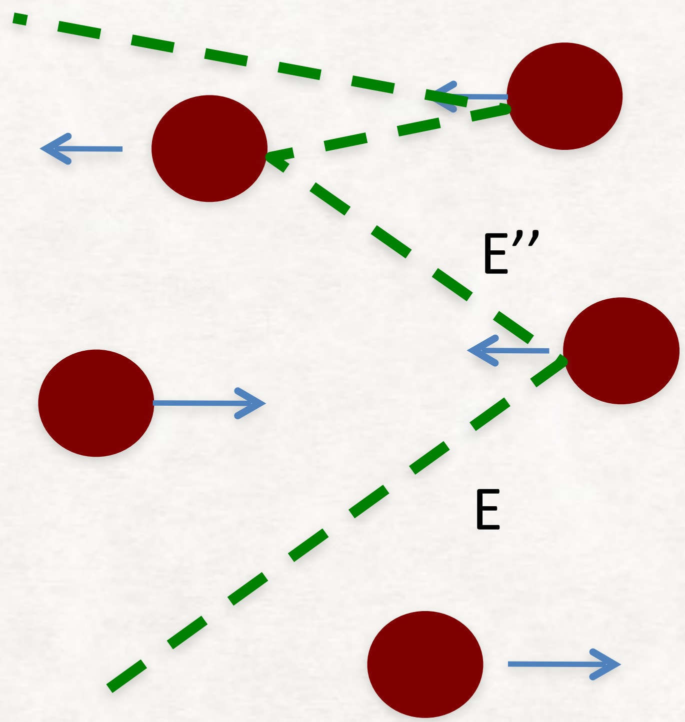
$$\Delta p \sim p \frac{v_A}{c}$$

In a time T which is the diffusion time as found in the last lecture. It follows that

$$D_{pp} = \langle \frac{\Delta P \Delta p}{\Delta t} \rangle \sim p^2 \frac{1}{T} \left(\frac{v_A}{c} \right)^2 \rightarrow \tau_{pp} = \frac{p^2}{D_{pp}} T \left(\frac{c}{v_A} \right)^2 \gg T$$

THE MOMENTUM CHANGE IS A SECOND ORDER PHENOMENON !!!

SECOND ORDER FERMI ACCELERATION



We inject a particle with energy E. In the reference frame of a cloud moving with speed b **the particle energy is:**

$$E' = \gamma E + \beta\gamma p\mu$$

and the momentum along x is:

$$p'_x = \beta\gamma E + \gamma p\mu$$

Assuming that the cloud is very massive compared with the particle, we can assume that the cloud is unaffected by the scattering, therefore the particle energy in the cloud frame does not change and the momentum along x is simply inverted, so that after 'scattering' $p'_x \rightarrow -p'_x$. The final energy in the Lab frame is therefore:

$$E'' = \gamma E' + \beta\gamma p'_x =$$

$$\gamma^2 E\left(1 + \beta^2 + 2\beta\mu\frac{p}{E}\right)$$

$$\frac{p}{E} = \frac{mv\gamma}{m\gamma} = v$$

Where v is now the dimensionless Particle velocity

It follows that:

$$E" = \gamma^2 E \left(1 + \beta^2 + 2\beta\mu v\right)$$

and:

$$\frac{E" - E}{E} = \gamma^2 \left(1 + 2\beta v\mu + \beta^2\right) - 1$$

and finally, taking the limit of non-relativistic clouds g→1:

$$\frac{E" - E}{E} \approx 2\beta^2 + 2\beta v\mu$$

We can see that the fractional energy change can be both positive or negative, which means that particles can either gain or lose energy, depending on whether the particle-cloud scattering is head-on or tail-on.

We need to calculate the probability that a scattering occurs head-on or Tail-on. The scattering probability along direction m **is proportional to the Relative velocity in that direction:**

$$P(\mu) = Av_{rel} = A\frac{\beta\mu + v}{1 + v\beta\mu} \rightarrow_{v\rightarrow 1} \approx A(1+\beta\mu)$$

The condition of normalization to unity:

$$\int_{-1}^{1} P(\mu) d\mu = 1$$

leads to A=1/2. It follows that the mean fractional energy change is:

$$\langle\frac{\Delta E}{E}\rangle = \int_{-1}^{1} d\mu P(\mu) \left(2\beta^2 + 2\beta\mu\right) = \frac{8}{3}\beta^2$$

NOTE THAT IF WE DID NOT ASSUME RIGID REFLECTION AT EACH CLOUD BUT RATHER ISOTROPIZATION OF THE PITCH ANGLE IN EACH CLOUD, THEN WE WOULD HAVE OBTAINED (4/3) b2 **INSTEAD OF (8/3)** b2

THE FRACTIONAL CHANGE IS A SECOND ORDER QUANTITY IN $\beta<<1$. This is the reason for the name SECOND ORDER FERMI ACCELERATION

The acceleration process can in fact be shown to become more Important in the relativistic regime where $\beta \rightarrow 1$

THE PHYSICAL ESSENCE CONTAINED IN THIS SECOND ORDER DEPENDENCE IS THAT IN EACH PARTICLE-CLOUD SCATTERING THE ENERGY OF THE PARTICLE CAN EITHER INCREASE OR DECREASE → WE ARE LOOKING AT A PROCESS OF DIFFUSION IN MOMENTUM SPACE

THE REASON WHY ON AVERAGE THE MEAN ENERGY INCREASES IS THAT HEAD-ON COLLISIONS ARE MORE PROBABLE THAN TAIL-ON COLLISIONS

WHAT IS DOING THE WORK?

We just found that particles propagating in a magnetic field can change their momentum (in modulus and direction)...

BUT MAGNETIC FIELDS CANNOT CHANGE THE MOMENTUM MODULUS... ONLY ELECTRIC FIELDS CAN

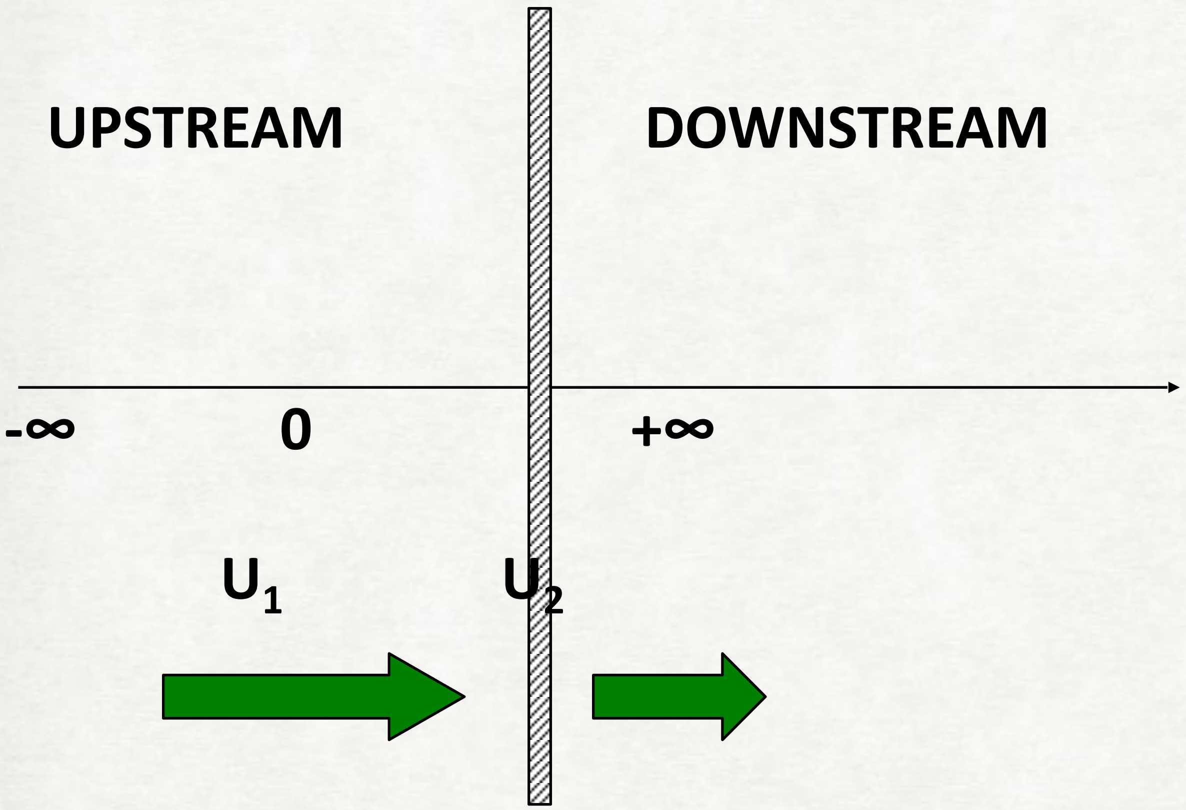
WHAT IS THE SOURCE OF THE ELECTRIC FIELDS???
Moving Magnetic Fields

The induced electric field is responsible for this first instance of particle acceleration

The scattering leads to momentum transfer, but to WHAT?

Recall that particles isotropize in the reference frame of the waves…

SHOCK SOLUTIONS



Let us sit in the reference frame in which the shock is at rest and look for stationary solutions

$$\frac{\partial}{\partial x}(\rho u) = 0$$

$$\frac{\partial}{\partial x}(\rho u^2 + P) = 0$$

$$\frac{\partial}{\partial x}\left(\frac{1}{2}\rho u^3 + \frac{\gamma}{\gamma - 1}uP\right) = 0$$

It is easy to show that aside from the trivial solution in which all quantities remain spatially constant, there is a discontinuous solution:

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2 + 2}$$

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1}$$

M_1 is the upstream Fluid Mach number

$$\frac{T_2}{T_1} = \frac{[2\gamma M_1^2 - \gamma(\gamma-1)][(\gamma-1)M_1^2 + 2]}{(\gamma+1)^2 M_1^2}$$

STRONG SHOCKS $M_1 >> 1$

In the limit of strong shock fronts these expressions get substantially simpler and one has:

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{\gamma+1}{\gamma-1}$$

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2}{\gamma+1}$$

$$\frac{T_2}{T_1} = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} M_1^2, \quad T_2 = 2\frac{\gamma-1}{(\gamma+1)^2} m u_1^2$$

ONE CAN SEE THAT SHOCKS BEHAVE AS VERY EFFICENT HEATING MACHINES IN THAT A LARGE FRACTION OF THE INCOMING RAM PRESSURE IS CONVERTED TO INTERNAL ENERGY OF THE GAS BEHIND THE SHOCK FRONT...

COLLISIONLESS SHOCKS

While shocks in the terrestrial environment are mediated by particle-particle collisions, astrophysical shocks are almost always of a different nature. The pathlength for ionized plasmas is of the order of:

$$\lambda \simeq \frac{1}{n\sigma} = 3.2 Mpc\, n_1^{-1} \left(\frac{\sigma}{10^{-25} cm^2} \right)^{-1}$$

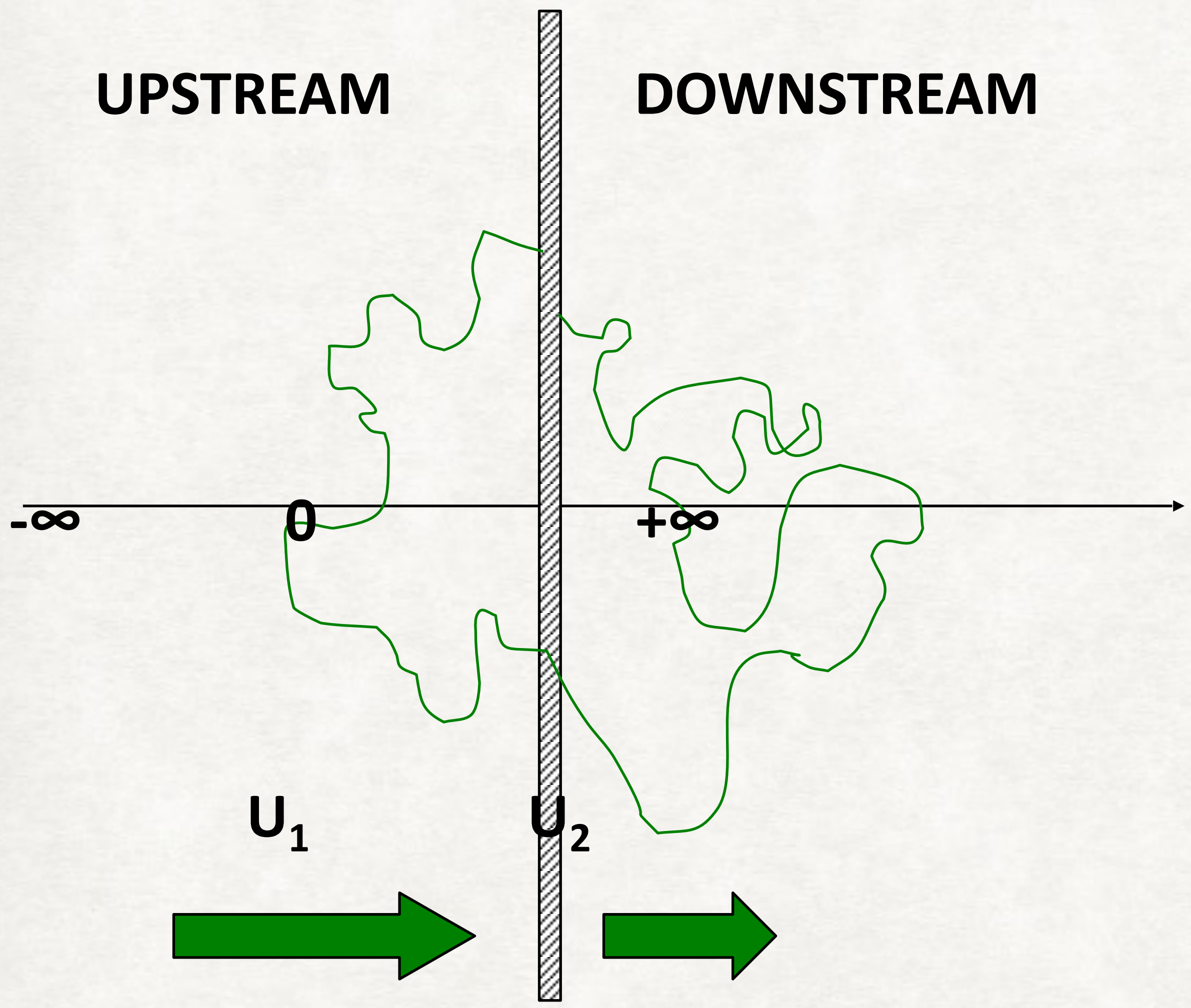
Absurdly large compared with any reasonable length scale. It follows that astrophysical shocks can hardly form because of particle-particle scattering but REQUIRE the mediation of magnetic fields. In the downstream gas the Larmor radius of particles is:

$$r_{L,th} \approx 10^{10} B_\mu T_8^{1/2}\ cm$$

The slowing down of the incoming flow and its isotropization (thermalization) is due to the action of magnetic fields in the shock region (COLLISIONLESS SHOCKS)

DIFFUSIVE SHOCK ACCELERATION
OR
FIRST ORDER FERMI ACCELERATION

BOUNCING BETWEEN APPROACHING MAGNETIC MIRRORS



Let us take a relativistic particle with energy E~p upstream of the shock. In the downstream frame:

$$E_d = \gamma E(1+\beta\mu) \quad 0 \leq \mu \leq 1$$

where β = u_1-u_2>0. In the downstream frame the direction of motion of the particle is isotropized and reapproaches the shock with the same energy but pitch angle μ'

$$E_u = \gamma E_d - \beta E_d \gamma \mu' = \gamma^2 E(1+\beta\mu)(1-\beta\mu')$$

$$-1 \leq \mu' \leq 0$$

In the non-relativistic case the particle distribution is, at zeroth order, isotropic
Therefore:

TOTAL FLUX

$$J = \int_0^1 d\Omega \frac{N}{4\pi} v\mu = \frac{Nv}{4} \quad \Rightarrow \quad P(\mu)d\mu = \frac{ANv\mu}{\frac{Nv}{4}} d\mu = 2\mu d\mu$$

The mean value of the energy change is therefore:

$$\langle \frac{E_u - E}{E} \rangle = -\int_0^1 d\mu 2\mu \int_{-1}^0 d\mu' 2\mu' \left[\gamma^2(1+\beta\mu)(1-\beta\mu') - 1\right] \approx \frac{4}{3}\beta = \frac{4}{3}(u_1 - u_2)$$

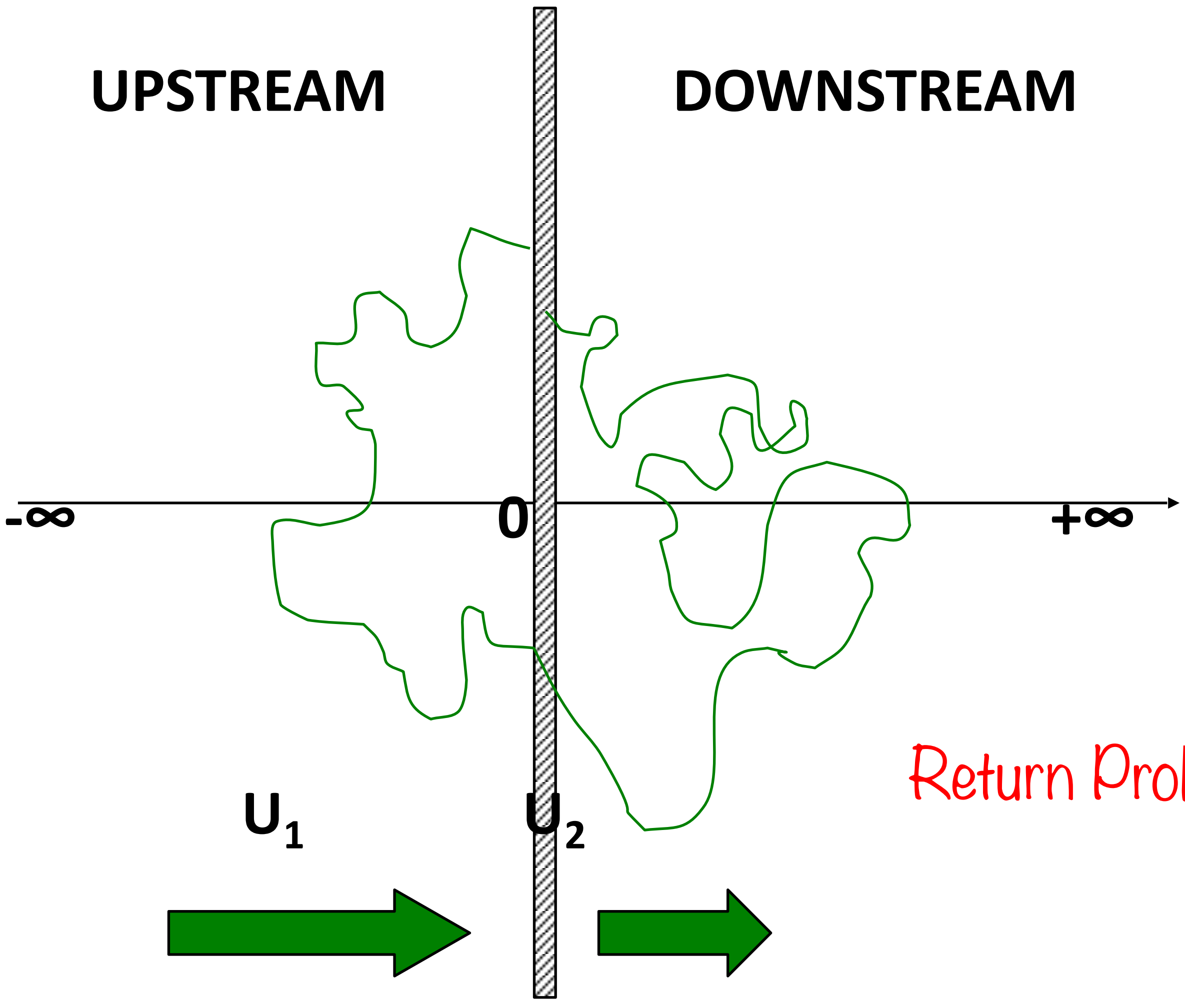
A FEW IMPORTANT POINTS:

I. **There are no configurations that lead to losses**

II. **The mean energy gain is now first order in β**

III. **The energy gain is basically independent of any detail on how particles scatter back and forth!**

RETURN PROBABILITIES AND SPECTRUM OF ACCELERATED PARTICLES



$$\varphi_{in} = \int_{-u_2}^{1} d\mu\, f_0(u_2 + \mu) = \frac{1}{2}(1 + u_2)^2$$

$$\varphi_{out} = \int_{-1}^{-u_2} d\mu\, f_0(u_2 + \mu) = \frac{1}{2}(1 - u_2)^2$$

Return Probability from Downstream

$$P_d = \frac{\varphi_{out}}{\varphi_{in}} = \frac{(1 - u_2)^2}{(1 + u_2)^2} \approx 1 - 4u_2$$

HIGH PROBABILITY OF RETURN FROM DOWNSTREAM BUT TENDS TO ZERO FOR HIGH U_2

ENERGY GAIN:

$$E_{k+1} = \left(1 + \frac{4}{3}V\right)E_k$$

$E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow$ --- $\rightarrow E_K=[1+(4/3)V]^K$ E0

$$\ln\left(\frac{E_K}{E_0}\right) = K \ln\left(1 + \frac{4}{3}(U_1 - U_2)\right)$$

$N_0 \rightarrow N_1=N_0\ P_{ret} \rightarrow$ --- $\rightarrow N_K=N_0\ P_{ret}^K$

$$\ln\left(\frac{N_K}{N_0}\right) = K \ln(1 - 4U_2)$$

Putting these two expressions together we get:

$$K = \frac{\ln\left[\frac{N_K}{N_0}\right]}{\ln\left[1-4U_2\right]} = \frac{\ln\left[\frac{E_K}{E_0}\right]}{\ln\left[1+\frac{4}{3}(U_1-U_2)\right]}$$

Therefore, after expanding for U<<1:

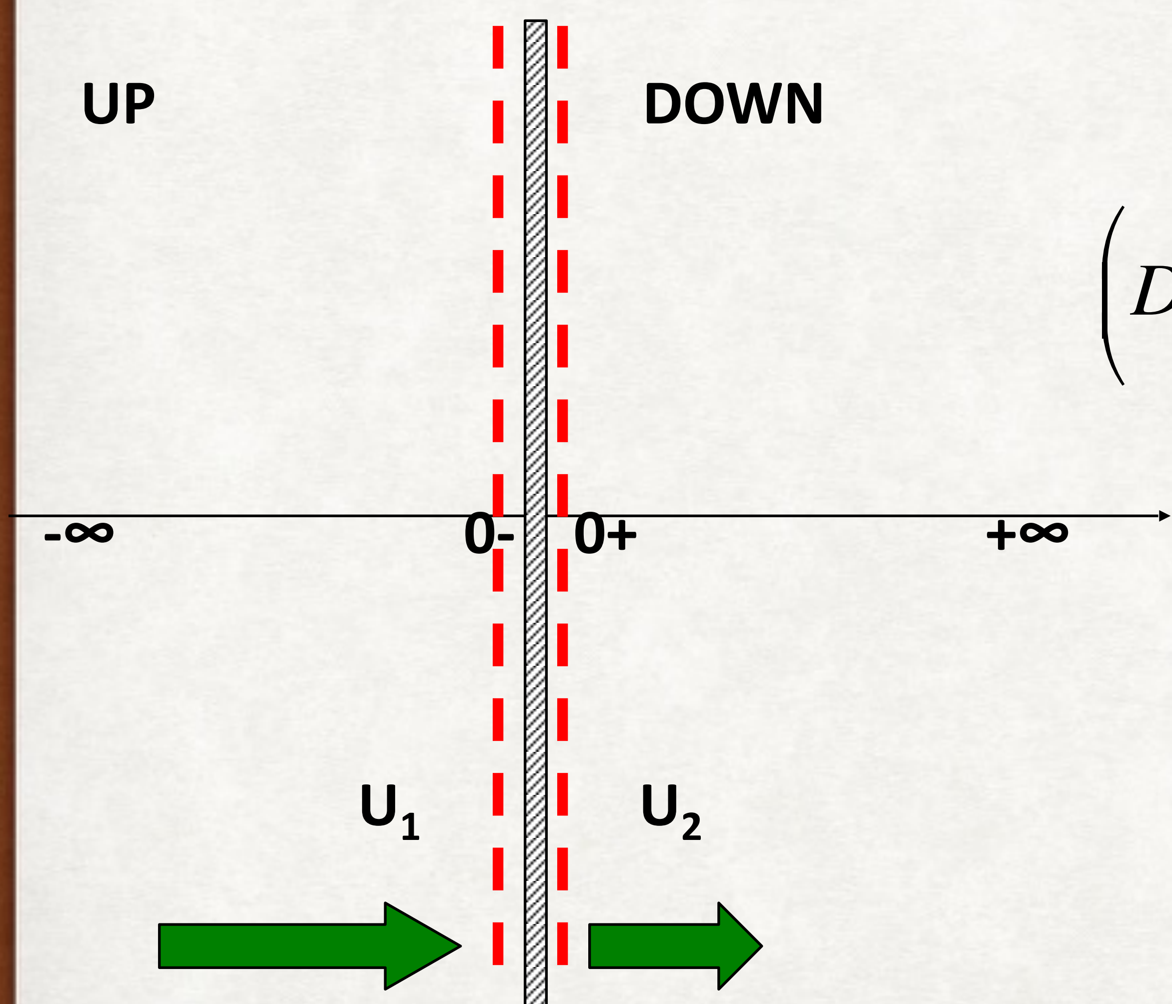
$$N(>E_K) = N_0\left(\frac{E_K}{E_0}\right)^{-\gamma} \qquad \gamma = \frac{3}{r-1} \qquad r = \frac{U_1}{U_2}$$

THE SLOPE OF THE DIFFERENTIAL SPECTRUM WILL BE **γ+1=(r+2)/(r-1) → 2 FOR r→4** (STRONG SHOCK)

THE TRANSPORT EQUATION APPROACH

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\left[D\frac{\partial f}{\partial x}\right] - u\frac{\partial f}{\partial x} + \frac{1}{3}\frac{du}{dx}\,p\frac{\partial f}{\partial p} + Q(x,p,t)$$

DIFFUSION ADVECTION COMPRESSION INJECTION



Integrating around the shock:

$$\left(D\frac{\partial f}{\partial x}\right)_2 - \left(D\frac{\partial f}{\partial x}\right)_1 + \frac{1}{3}(u_2 - u_1)\,p\frac{df_0(p)}{dp} + Q_0(p) = 0$$

Integrating from upstr. infinity to 0-:

$$\left(D\frac{\partial f}{\partial x}\right)_1 = u_1 f_0$$

and requiring homogeneity downstream:

$$p\frac{df_0}{dp} = \frac{3}{u_2 - u_1}\left(u_1 f_0 - Q_0\right)$$

THE TRANSPORT EQUATION APPROACH

INTEGRATION OF THIS SIMPLE EQUATION GIVES:

$$f_0(p) = \frac{3u_1}{u_1 - u_2} \frac{N_{inj}}{4\pi p_{inj}^2} \left(\frac{p}{p_{inj}} \right)^{\frac{-3u_1}{u_1 - u_2}}$$

DEFINE THE COMPRESSION FACTOR
$r = u_1/u_2 \rightarrow 4$ (strong shock)

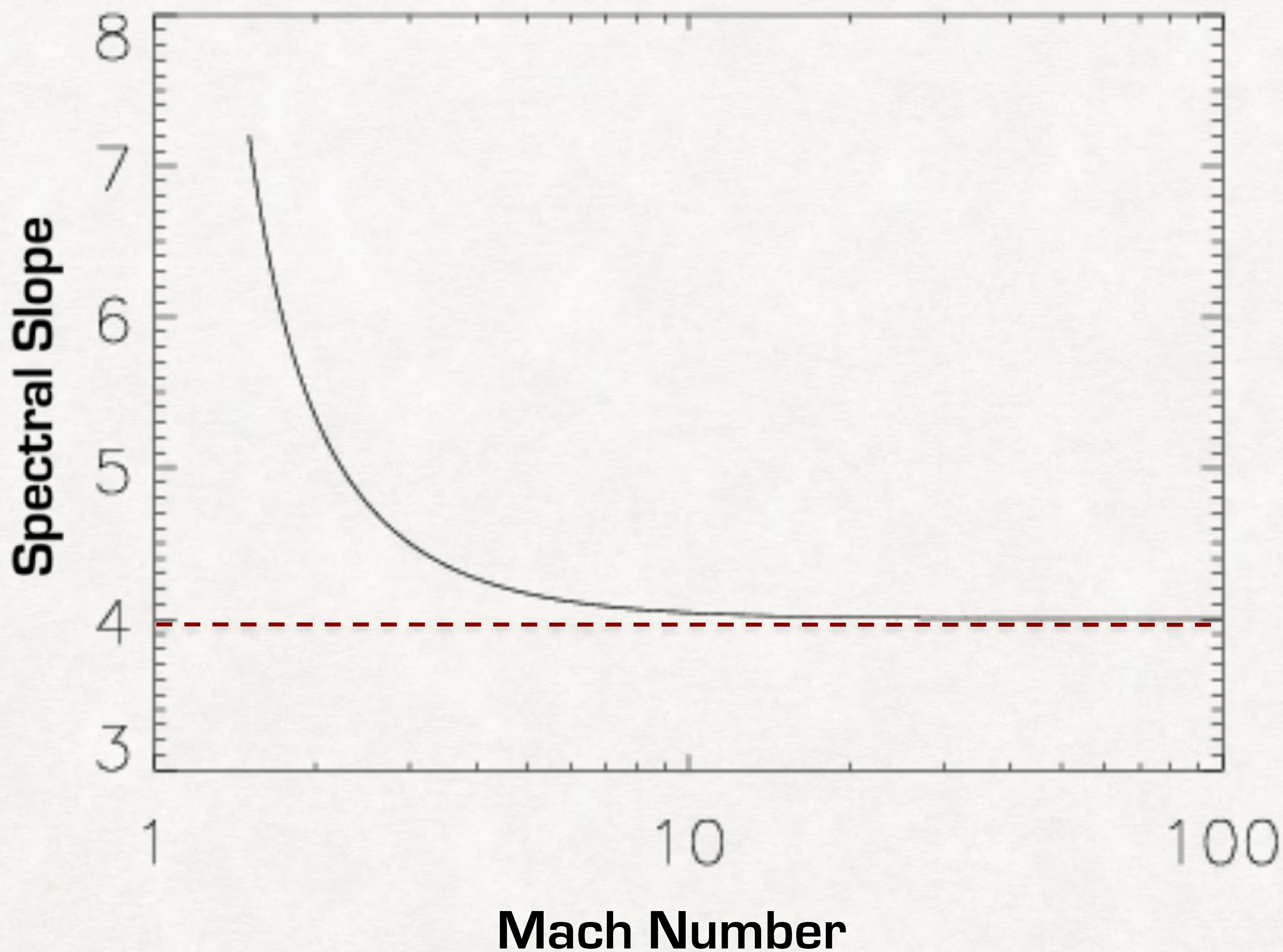
THE SLOPE OF THE SPECTRUM IS

$$\frac{3u_1}{u_1 - u_2} = \frac{3}{1 - 1/r} \rightarrow 4 \quad \text{if r} \rightarrow 4$$

NOTICE THAT: $N(p)dp = 4\pi p^2 f(p)dp \rightarrow N(p) \propto p^{-2}$

1. **THE SPECTRUM OF ACCELERATED PARTICLES IS A POWER LAW IN MOMENTUM EXTENDING TO INFINITE MOMENTA**
2. **THE SLOPE DEPENDS UNIQUELY ON THE COMPRESSION FACTOR AND IS INDEPENDENT OF THE DIFFUSION PROPERTIES**
3. **INJECTION IS TREATED AS A FREE PARAMETER WHICH DETERMINES THE NORMALIZATION**

TEST PARTICLE SPECTRUM



SOME IMPORTANT COMMENTS

- THE STATIONARY PROBLEM DOES NOT ALLOW TO HAVE A MAX MOMENTUM!
- THE NORMALIZATION IS ARBITRARY THEREFORE THERE IS NO CONTROL ON THE AMOUNT OF ENERGY IN CR
- AND YET IT HAS BEEN OBTAINED IN THE TEST PARTICLE APPROXIMATION
- THE SOLUTION DOES NOT DEPEND ON WHAT IS THE MECHANISM THAT CAUSES PARTICLES TO BOUNCE BACK AND FORTH
- FOR STRONG SHOCKS THE SPECTRUM IS UNIVERSAL AND CLOSE TO E^{-2}
- IT HAS BEEN IMPLICITELY ASSUMED THAT WHATEVER SCATTERS THE PARTICLES IS AT REST (OR SLOW) IN THE FLUID FRAME

MAXIMUM ENERGY

The maximum energy in an accelerator is determined by either the age of the accelerator compared with the acceleration time or the size of the system compared with the diffusion length D(E)/u. The hardest condition is the one that dominates.

Using the diffusion coefficient in the ISM derived from the B/C ratio:

$$D(E) \approx 3 \times 10^{28} E_{GeV}^{1/3} cm^2/s$$

and the velocity of a SNR shock as u=5000 km/s one sees that:

$$t_{acc} \sim D(E)/u^2 \sim 4 \times 10^3 E_{GeV}^{1/3} \ years$$

Too long for any useful acceleration → NEED FOR ADDITIONAL TURBULENCE

$$t_{acc}(p) = \langle t \rangle = \frac{3}{u_1 - u_2} \int_{p_0}^{p} \frac{dp'}{p'} \left[\frac{D_1(p')}{u_1} + \frac{D_2(p')}{u_2} \right]$$

ENERGY LOSSES AND ELECTRONS

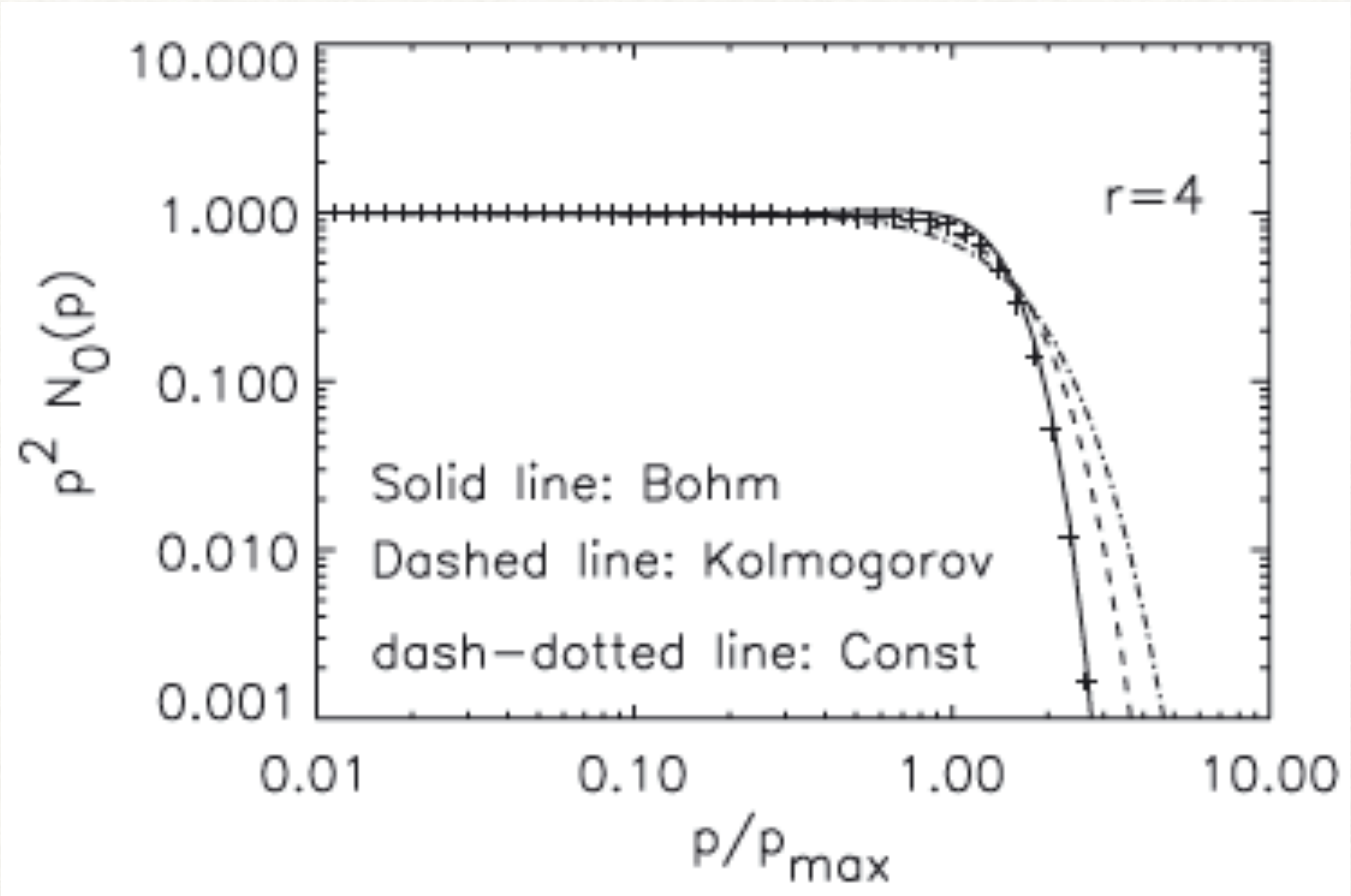
For electrons, energy losses make acceleration even harder.

The maximum energy of electrons is determined by the condition:

$$t_{acc} \leq Min\left[Age, \tau_{loss}\right]$$

Where the losses are mainly due to synchrotron and inverse Compton Scattering.

ELECTRONS IN ONE SLIDE



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APPENDIX 1 HYDRODYNAMICS AND SHOCKS

There are many instances of astrophysical systems that result in explosive phenomena in which large amounts of mass and energy are released in the surrounding medium (interstellar medium or intergalactic medium) at high speed. The ejected material behaves as a fluid, though often the importance of magnetic fields cannot be neglected.

Here I will discuss the basic laws that govern the dynamics of such a fluid, under ideal conditions in which the fluid evolves adiabatically and the effects of thermal conductivity can be neglected.

I will show how the laws that govern the motion of such a fluid lead to conclude that in some conditions shock waves can develop in the fluid.

These concepts are of particular importance in supernova explosions, which Are likely to play an important role for particle acceleration in the universe.

I will restrict the attention to fluid that move subrelativistically, so that only Newtonian dynamics applies.

I will also comment upon the collisionless nature of the shock waves that develop in astrophysics (with some exceptions).

CONSERVATION OF MASS

Let us consider a fixed infinitesimal volume dV where the matter density is r. The mass in the volume rdV remains constant unless mass is allowed to Flow in and out of the volume dV. The total mass is

$$\int \rho dV$$

and changes in time because of the flux of mass per unit time and volume across the surface dA that surrounds dV:

$$-\frac{d}{dt}\int \rho dV = \oint \rho\vec{v}\cdot d\vec{A} \equiv \int \nabla\cdot(\rho\vec{v})dV$$

Gauss Theorem

It follows that:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$$

CONSERVATION OF MOMENTUM

An element of surface suffers a pressure p and a force over the volume:

$$-\oint_{d\vec{A}} p d\vec{A} = -\int_{dV} \nabla p dV$$

The force exerted on the fluid element of mass ρdV is:

$$\rho dV \frac{D\vec{v}}{Dt} = -\nabla p dV \rightarrow \rho \frac{D\vec{v}}{Dt} = -\nabla p$$

Where D/Dt is the convective derivative. Let us consider a fluid element that is at x at time t and moves with velocity v(x,t). At time t+dt the fluid element is located at x+vdt, therefore the acceleration is

$$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}$$

It follows that:

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{\nabla p}{\rho}$$

CONSERVATION OF ENERGY

In the assumption of adiabatic evolution of the fluid, the entropy per unit mass s is conserved:

$$\frac{\partial s}{\partial t} + \vec{v} \cdot \nabla s = 0$$

and using conservation of mass, one immediately gets:

$$\frac{\partial (\rho s)}{\partial t} + \nabla \cdot (\rho s \vec{v}) = 0$$

Introducing the specific enthalpy: w=e+p/ρ, one can write:

$$dw = T ds + \frac{dp}{\rho} = \frac{dp}{\rho}$$

Adiabatic→ds=0

Which leads to:

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{\nabla p}{\rho} = -\nabla \left(e + \frac{p}{\rho} \right)$$

For a polytropic gas with adiabatic index g one has that the energy density per Unit volume is u=p/(g−1) therefore:

$$u = \rho e = \frac{p}{\gamma - 1} \rightarrow e = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

So that

$$w = e + \frac{p}{\rho} = \frac{\gamma}{\gamma - 1}\frac{p}{\rho}$$

And the previous equation becomes:

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{\gamma}{\gamma - 1}\nabla\left(\frac{p}{\rho}\right)$$

In the one dimensional stationary case one has

$$v\frac{\partial v}{\partial x} + \frac{\gamma}{\gamma - 1}\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$$

And using the eqn for conservation of mass one immediately gets:

$$\frac{\partial}{\partial x}\left[\frac{1}{2}\rho v^3 + \frac{\gamma}{\gamma - 1}vp\right] = 0 \rightarrow \frac{1}{2}\rho v^3 + \frac{\gamma}{\gamma - 1}vp = Constant$$

APPENDIX 2: ACCELERATION TIME

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\left[D\frac{\partial f}{\partial x}\right] - u\frac{\partial f}{\partial x} + \frac{1}{3}\frac{du}{dx}p\frac{\partial f}{\partial p} + Q(x,p,t)$$

Let us move to the Laplace transform:

$$g(s,x,p) = \int_0^\infty dt e^{-st} f$$

so that the transport equation becomes:

$$sg + u\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left[D\frac{\partial g}{\partial x}\right] + \frac{1}{3}\left(\frac{du}{dx}\right)p\frac{\partial g}{\partial p} + \frac{Q(x,p)}{s}$$

Integrating this equation between x=0$^-$ and x=0$^+$ one gets:

$$0 = \left[D\frac{\partial g}{\partial x}\right]_2 - \left[D\frac{\partial g}{\partial x}\right]_1 + \frac{1}{3}(u_2 - u_1)p\frac{\partial g_0}{\partial p} + \frac{Q_0(p)}{s}$$

Where we have assumed that:

$$Q(x,p) = Q_0(p)\delta(x) \quad and \quad Q_0(p) = A\delta(p - p_{inj})$$

UPSTREAM

$$s g_1 + u_1 \frac{\partial g_1}{\partial x} = \frac{\partial}{\partial x}\left[D_1 \frac{\partial g_1}{\partial x}\right]$$

and assuming that the diffusion coefficient is independent upon location x
the solution has the form:

$$g_1(s,p,x) = g_0(s,p)\exp\left[\beta_1 x\right] \qquad x < 0$$

$$\beta_1 = \frac{u_1 + \sqrt{u_1^2 + 4D_1 s}}{2D_1} = \frac{u_1}{2D_1}\left[1 + \sqrt{1 + \frac{4D_1 s}{u_1^2}}\right]$$

$$\left[D_1 \frac{\partial g_1}{\partial x}\right]_1 = \frac{u_1}{2}\left[1 + \sqrt{1 + \frac{4D_1 s}{u_1^2}}\right] g_0 = \sigma_1 g_0$$

DOWNSTREAM

$$s g_2 + u_2 \frac{\partial g_2}{\partial x} = \frac{\partial}{\partial x}\left[D_2 \frac{\partial g_2}{\partial x}\right]$$

Proceeding as in the previous case:

$$g_2(s,p,x) = g_0(s,p)\exp\left[\beta_2 x\right] \qquad x > 0$$

$$\beta_2 = \frac{u_2 - \sqrt{u_2^2 + 4D_2 s}}{2D_2} = \frac{u_2}{2D_2}\left[1 - \sqrt{1 + \frac{4D_2 s}{u_2^2}}\right]$$

$$\left[D_2 \frac{\partial g_2}{\partial x}\right]_2 = \frac{u_2}{2}\left[1 - \sqrt{1 + \frac{4D_2 s}{u_2^2}}\right] g_0 = \sigma_2 g_0$$

Notice that in the long time limit, namely s→0 one gets the well known result:

$$\left[D_1 \frac{\partial g_1}{\partial x}\right]_1 = u_1 g_0 \qquad \left[D_2 \frac{\partial g_2}{\partial x}\right]_2 = 0$$

Notice that one can easily write:

$$\left[D_1 \frac{\partial g_1}{\partial x}\right]_1 = \frac{u_1}{2}\left[1+\sqrt{1+\frac{4D_1 s}{u_1^2}}\right] g_0 - u_1 g_0 + u_1 g_0 = -\frac{u_1}{2} g_0 + \frac{u_1}{2}\sqrt{1+\frac{4D_1 s}{u_1^2}} g_0 + u_1 g_0 =$$

$$= g_0 u_1 A_1 + u_1 g_0 \qquad A_1 = \frac{1}{2}\left[-1+\sqrt{1+\frac{4D_1 s}{u_1^2}}\right]$$

In this way A_1 has the same property as σ_1 namely they both vanish in the long time limit $s \to 0$.

Substituting in the equation at the shock:

$$0 = \left[D \frac{\partial g}{\partial x}\right]_2 - \left[D \frac{\partial g}{\partial x}\right]_1 + \frac{1}{3}(u_2 - u_1)\, p \frac{\partial g_0}{\partial p} + \frac{Q_0(p)}{s}$$

$$\Downarrow$$

$$(\sigma_2 - u_1 A_1 - u_1)\, g_0 + \frac{1}{3}(u_2 - u_1) p \frac{\partial g_0}{\partial p} + \frac{Q_0(p)}{s} = 0$$

The homogeneous equation associated with this is:

$$(\sigma_2 - u_1 A_1 - u_1)\,\tilde{g}_0 + \frac{1}{3}(u_2 - u_1)p\frac{\partial \tilde{g}_0}{\partial p} = 0$$

Which has the solution:

$$\tilde{g}_0 = \exp\left[\int_{p_0}^{p} \frac{dp'}{p'}\frac{3}{u_1 - u_2}(\sigma_2 - u_1 A_1 - u_1)\right] = \left(\frac{p}{p_0}\right)^{-\frac{3u_1}{u_1 - u_2}} \exp\left[\int_{p_0}^{p} \frac{dp'}{p'}\frac{3}{u_1 - u_2}(\sigma_2 - u_1 A_1)\right]$$

The general solution of the equation has the form:

$$g_0 = \tilde{g}_0 \lambda$$

therefore the equation for l must be:

$$\frac{1}{3}(u_2 - u_1)p\frac{\partial \lambda}{\partial p}\tilde{g}_0 + \frac{Q_0}{s} = 0$$

which is readily solved:

$$\lambda = \int_{p_0}^{p} \frac{dp'}{p'}\frac{3}{u_1 - u_2}\frac{A}{s}\delta(p' - p_0)\left(\frac{p'}{p_0}\right)^{\frac{3u_1}{u_1 - u_2}} \exp\left[-\int_{p_0}^{p'} \frac{dp"}{p"}\frac{3}{u_1 - u_2}(\sigma_2 - u_1 A_1)\right] =$$

$$= \frac{A}{s}\frac{3}{u_1 - u_2}$$

It follows that the solution of our equation is:

$$g_0 = \frac{3A}{s(u_1 - u_2)} \left(\frac{p}{p_0}\right)^{-\frac{3u_1}{u_1-u_2}} \exp\left[\int_{p_0}^{p} \frac{dp'}{p'} \frac{3}{u_1 - u_2}(\sigma_2 - u_1 A_1)\right]$$

and carrying out the Laplace inverse transform:

$$f_0(t,p) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} ds\ e^{ts} g_0(s,p)$$

Note that the pole in the equation for g_0 is at s=0 and it is obvious that in the limit of large times one has:

$$f_0(p) = \frac{3A}{s(u_1 - u_2)} \left(\frac{p}{p_0}\right)^{-\frac{3u_1}{u_1-u_2}} \equiv K(p)$$

Therefore one can write:

$$f_0(p,t) = K(p)\exp\left[h(p,s)\right]$$

Let us introduce the function:

$$\tilde{f}(p,t) = K(p)\int_0^t dt'\phi(p,t') \quad with$$

$$\phi(p,t) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} ds\ e^{ts}\exp\left[h(p,s)\right]$$

Taking the Laplace Transform of this new function one has:

$$\int_0^\infty dt\ e^{-st}K(p)\int_0^t dt'\phi(p,t') = \frac{K(p)}{s}\exp\left[h(p,s)\right] = g_0(p,s)$$

This means that the solution of our problem is the spectrum K(p) and infinite time times a probability function that at time t one can have a particle with momentum p. Indeed one has that:

$$\int_0^\infty dt\ \phi(p,t') = 1$$

Namely the function is correctly normalized.

One can now use the obvious property that:

$$\int_0^\infty \mathrm{d}t\, \phi(p,t)\, \mathrm{e}^{-ts} = \exp[h(p,s)]$$

From which it follows that the average time to get particles with momentum p is:

$$\langle t \rangle = -\left[\frac{\partial h}{\partial s}\right]_{s=0}$$

It follows that:

$$t_{acc}(p) = \langle t \rangle = \frac{3}{u_1 - u_2} \int_{p_0}^{p} \frac{dp'}{p'} \left[\frac{D_1(p')}{u_1} + \frac{D_2(p')}{u_2}\right]$$