A MODEL FOR THE STUDY OF VERY-HIGH-ECCENTRICITY ASTEROIDAL MOTION: THE 3:1 RESONANCE

S.Ferraz-Mello and J.C.Klafke

Instituto Astronômico e Geofísico
Universidade de São Paulo
São Paulo

INTRODUCTION

Recent work by one of the authors (Ferraz-Mello, 1989, 1990 a,b) and by Morbidelli and Giorgilli (1990) has shown the existence of very-high-eccentricity stable and unstable equilibrium solutions (corotation centers) in the averaged Sun-Jupiter-asteroid planar problem, when a secular resonance and a resonance of periods occur simultaneously. They correspond to stationary motions in which the orbits of the asteroid and Jupiter share the same apsidal line. In the case of a 3:1 resonance of periods, two of these corotation centers form a stable-unstable pair at $e = 0.812$ and $e = 0.788$, respectively. The stable corotation center corresponds to a maximum of the energy ($E_S = -1.775728$ in astronomical units). For values close to this maximum the motions are regular oscillations in the neighbourhood of the corotation center (as those shown in the left-hand side of fig. 2).

The study of the phase portrait of such dynamical system may be done by means of purely numerical techniques, but, generally, the CPU times involved are large and limit the possible exploration of the phase space. More extended analyses become possible if an analytical averaging of the equations is made before the numerical integration. However, the classical technique for the expansion of the potential of the disturbing forces due to Jupiter, in terms of Keplerian elements, may be used only for small eccentricities; indeed, as shown by Sundmann (Silva, 1909; Hagihara, 1971), the convergence of the series is limited to $e \approx 0.33$ in the resonance $3:1$, to $e \approx 0.18$ in the resonance $2:1$ and only to $e \approx 0.09$ in the resonance $3:2$, values which are less than the observed ones (the original proof by Sundmann was restricted to orbits with circulating perihelia – but this is generally the case). In order to circumvent this difficulty, we may use asymmetric expansions (Ferraz-Mello, 1987; Ferraz-Mello and Sato, 1989); they are Taylor series about generic points in the phase space (with $e_0 \neq 0$) and may represent the disturbing potential for large values of e. However, there is no reason for which these series may have a better convergence than the classical, symmetric one – on the contrary, the convergence must become poor when $e_0 \to 1$. Thus, analytically, the asymmetric series can only be used for the study of librations and corotations of small amplitude, when the motion remains in the neighbourhood of the point about which the expansion is done.

Predictability, Stability, and Chaos in N-Body Dynamical Systems
Edited by A.E. Roy, Plenum Press, New York, 1991

THE MODEL

In the planar model presented in this paper, in order to cope with this difficulty, the phase space is divided into a given number of cells and the analytical averaging of the disturbing potential is done locally, in the center of each cell. The averaged equations are, then, integrated numerically and, at every point in the integration, the coefficients of the series representation of the disturbing function are taken as computed in the center of the corresponding cell.

The adopted equations are non-canonical. The reason for this choice is that the asymmetric expansion of Ferraz-Mello and Sato is done using Keplerian elements. These equations are obtained straightforwardly from Lagrange's equations for the variation of the elements:

$$\frac{da}{dt} = \frac{2r}{na}[h\frac{\partial F}{\partial k} - k\frac{\partial F}{\partial h} - \frac{\partial F}{\partial \sigma_1}] \tag{1}$$

$$\frac{d\sigma_1}{dt} = (r+1)n_1 - rn + \frac{2r}{na}\frac{\partial F}{\partial a} - \frac{r\beta}{na^2(1+\beta)}[k\frac{\partial F}{\partial k} + h\frac{\partial F}{\partial h}] \tag{2}$$

$$\frac{dk}{dt} = \frac{\beta}{na^2}\frac{\partial F}{\partial h} - h\frac{d\sigma_1}{dt} - \frac{k\beta}{2a(1+\beta)}\frac{da}{dt} \tag{3}$$

$$\frac{dh}{dt} = -\frac{\beta}{na^2}\frac{\partial F}{\partial k} + k\frac{d\sigma_1}{dt} - \frac{h\beta}{2a(1+\beta)}\frac{da}{dt}. \tag{4}$$

The variables are the semi-major axis a and the parameters k, h, σ_1 defined through the equations

$$\begin{aligned} k &= e\cos\sigma \\ h &= e\sin\sigma \\ \sigma &= \phi - \varpi \\ \sigma_1 &= \phi - \varpi_1 \\ \phi &= (r+1)\lambda_1 - r\lambda. \end{aligned} \tag{5}$$

ϕ is the critical angle associated with the q^{th}-order resonance $(p+q):p$ (or $(r+1):r$ with $r = p/q$). e, e_1 are the eccentricities, λ, λ_1 the mean longitudes, ϖ, ϖ_1 the longitudes of the perihelia, n, n_1 the mean motions and $\beta = \sqrt{1-e^2}$; the subscript 1 refers to Jupiter. The fifth equation, in the derivative of the mean synodic longitude $Q = \lambda - \lambda_1$, is not considered, since the function F is averaged over q times the mean synodic period and $\frac{\partial F}{\partial Q} = 0$. The orbital elements of Jupiter are assumed constant.

The function F is the averaged disturbing function:

$$\begin{aligned} F = \frac{\mu}{a_1}[&A_0 + A_1\delta k + A_2\delta h + \frac{1}{2}A_3\delta k^2 + \frac{1}{2}A_4\delta h^2 + A_5\delta k\delta h \\ &+(A_6 + A_8\delta k + A_{10}\delta h)e_1\cos\sigma_1 + (A_7 + A_9\delta k + A_{11}\delta h)e_1\sin\sigma_1 \\ &+ \frac{1}{2}A_{12}e_1^2 + \frac{1}{2}A_{13}e_1^2\cos 2\sigma_1 + \frac{1}{2}A_{14}e_1^2\sin 2\sigma_1] \end{aligned} \tag{6}$$

expanded about one center k_0, h_0 up to the second power of the differences $\delta k = k - k_0, \delta h = h - h_0$ (see Ferraz-Mello and Sato, 1989). The coefficients A_j are functions of the semi- major axis a and of the chosen center k_0, h_0.

The Lagrangian equations for a, k, h, σ_1 have the energy-like first integral

$$-E = \frac{\mu}{2a} + \frac{p+q}{p} n_1 n a^2 + F. \tag{7}$$

The value of the coefficients A_j and their derivatives with respect to a are calculated previously in a set of points k_0, h_0. These coefficients are assumed to allow a good representation of the function F in a small domain around k_0, h_0, in the neighbourhood of the value of the semi-major axis a characteristic of the given resonance. The plane k, h is then divided into a finite number of square cells. During the numerical integration of the equations, at each point, the expansion about the center of the cell where the point is found, is selected. The dependence of the coefficients with a is given by a linear approximation.

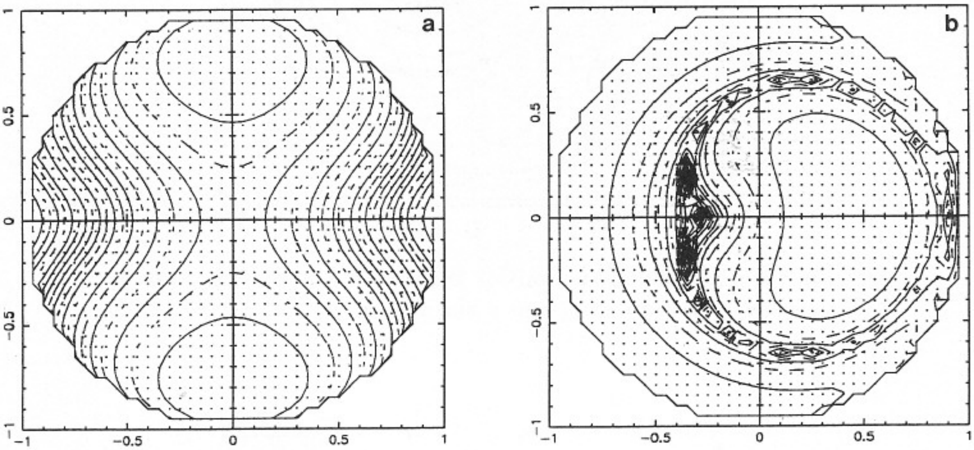

Fig. 1 – Level curves $A_0 = const.$ in the plane k, h. (a) Resonance 3:1. (b) Resonance 3:2. The dots are the centers used for the local expansions.

Figure 1 shows the level curves $A_0 = const.$ in the plane k, h, in two cases: the resonances 3:1 and 3:2. Some results for the resonance 3:1 are discussed in this paper. The level curves for the resonance 3:2 show some difficulties we face in this resonance, but not in the resonance 3:1. In the case 3:2 we may see a *reef* along the line corresponding to averages over orbits going through a collision with Jupiter. The infinite values of A_0 on this line, in fact, are not seen in the figure, since the function is sampled only in the points of a finite grid. In both cases the calculations were done in the center of a net of squares sized 0.05×0.05.

THE RESONANCE 3:1

In this section, some preliminary results concerning the resonance 3:1, obtained with this method, are given. Figures 2 and 3 show surfaces of section defined by $\sigma = \pi/2$ (with $\dot{\sigma} < 0$) and the energies (referred to the maximum) $\Delta E = -4.3 \times 10^{-5} E_S$ and $\Delta E = -4.7 \times 10^{-5} E_S$, respectively. These figures extend to high eccentricities results previously found by Wisdom (1983,1985) and by Henrard and Caranicolas (1990) (They are somewhat larger than those obtained using the classical Laplacian expansion). In both figures there is a stable periodic solution, near the origin, surrounded by

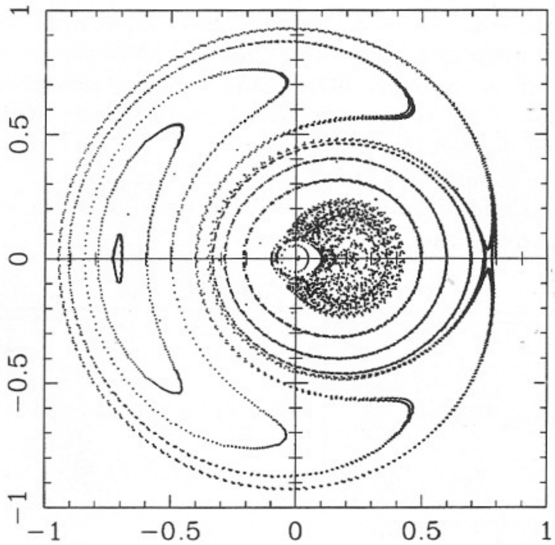

Fig. 2 – Surface of section for $\Delta E = -4.3 \times 10^{-5} E_S$. The axes are $\epsilon.\cos(\varpi - \varpi_1)$ and $\epsilon.\sin(\varpi - \varpi_1)$.

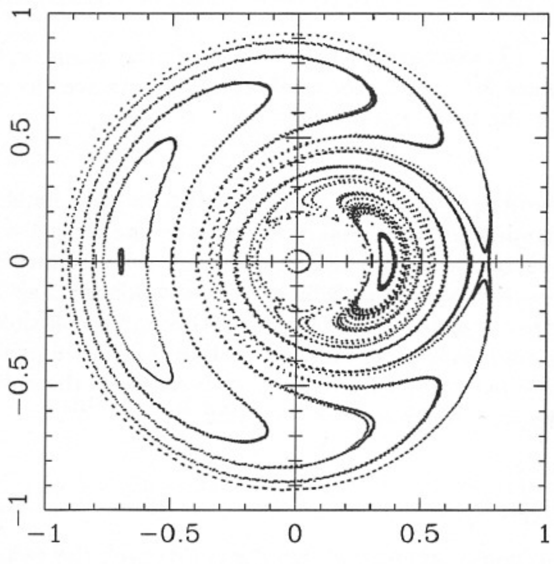

Fig. 3 – Surface of section for $\Delta E = -4.7 \times 10^{-5} E_S$. Axes as in fig. 2.

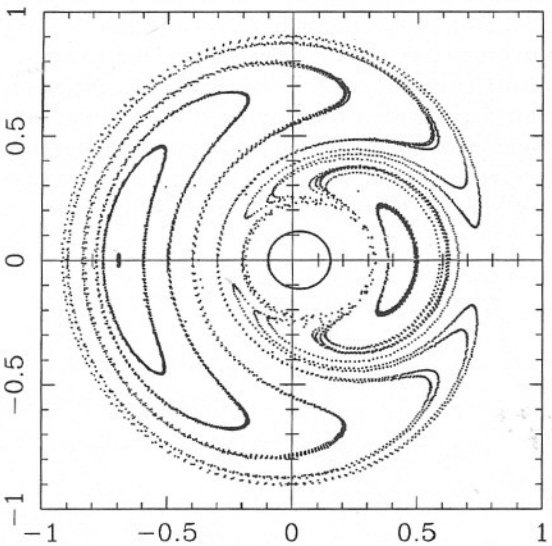

Fig. 4 – Surface of section for $\Delta E = -5.05 \times 10^{-5} E_S$. Axes as in fig. 2.

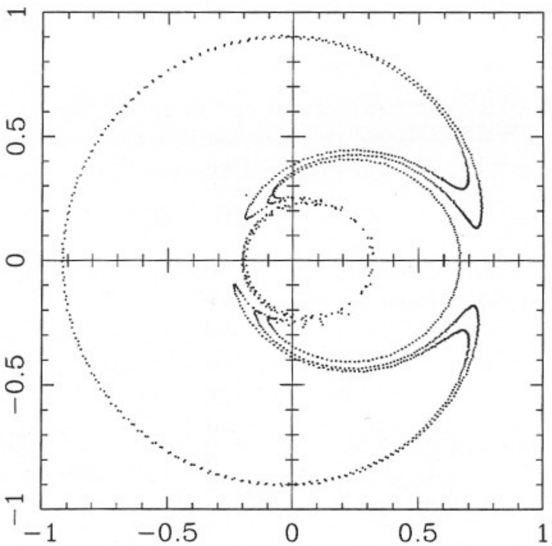

Fig. 5 – Detail of fig. 4. The chaotic solution making the communication between the inner and outer part of the plane of section.

regular motions. The boundary of this region (called *zone of uncertainty*, by Wisdom) is the inner limit of the chaotic region emanating from the saddle in the horizontal axis, at $k \approx -0.1$ (fig. 2) and $k \approx -0.2$ (fig. 3). In fig. 3 another center is seen, at $k \approx 0.35$, as well as some seemingly regular curves around it. In fig. 2 a similar center has not been found. In both cases, the whole inner region is enveloped by a bunch of regular curves. In the outer part of the phase space the topology is governed by the center at $k = -0.7$ and the saddle near $k = +0.8$. The motions about the center are regular. Some irregularities are seen starting from the saddle, but the precision of the calculations is not good enough to allow us to say that they mean chaos. Anyway, chaotic motion is expected there and the results obtained serve to say that the diffusion in such region must be very slow (every integration in the outer region covers 300,000 years).

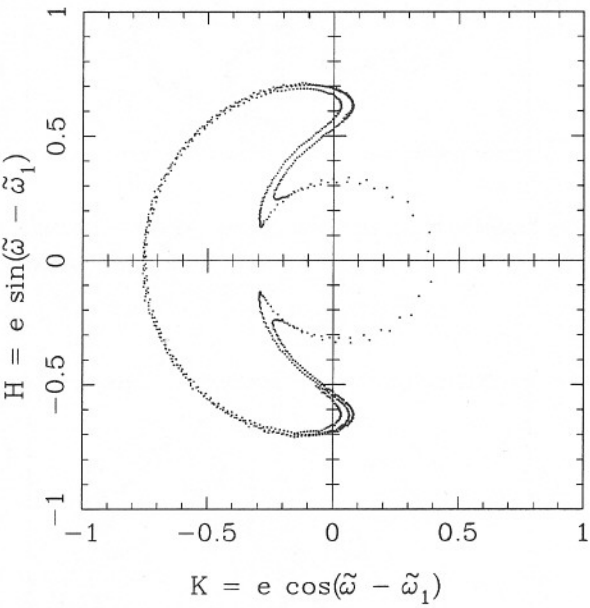

Fig. 6 – Solution corresponding to the present orbital elements of (887) *Alinda* for 300,000 years.

In the inner region of these figures we may see orbits allowing the eccentricity to vary between $0.1 - 0.2$ and $0.4 - 0.5$; in the outer one the allowed variation is between 0.3 and 0.9. However, these two regions, for these energies, do not communicate.

Figure 4 shows a similar surface of section for $\Delta E = -5.05 \times 10^{-5} E_S$. Now, the inner and outer regions communicate; the outer stable and unstable manifolds of the innermost saddle and the inner manifolds of the outermost one entangle. While the chaotic regions of figures 2 and 3 are associated with the existence of homoclinic points, the communication between the two regions is associated with heteroclinic points. One solution leading from $e < 0.2$ to $e = 0.9$ is, now, possible. The evolution shown in fig. 5 corresponds to 7×10^5 years. One asteroid (or meteoroid), in such a motion, would keep its eccentricity higher than 0.6 for periods of $\approx 10^5$ years, one

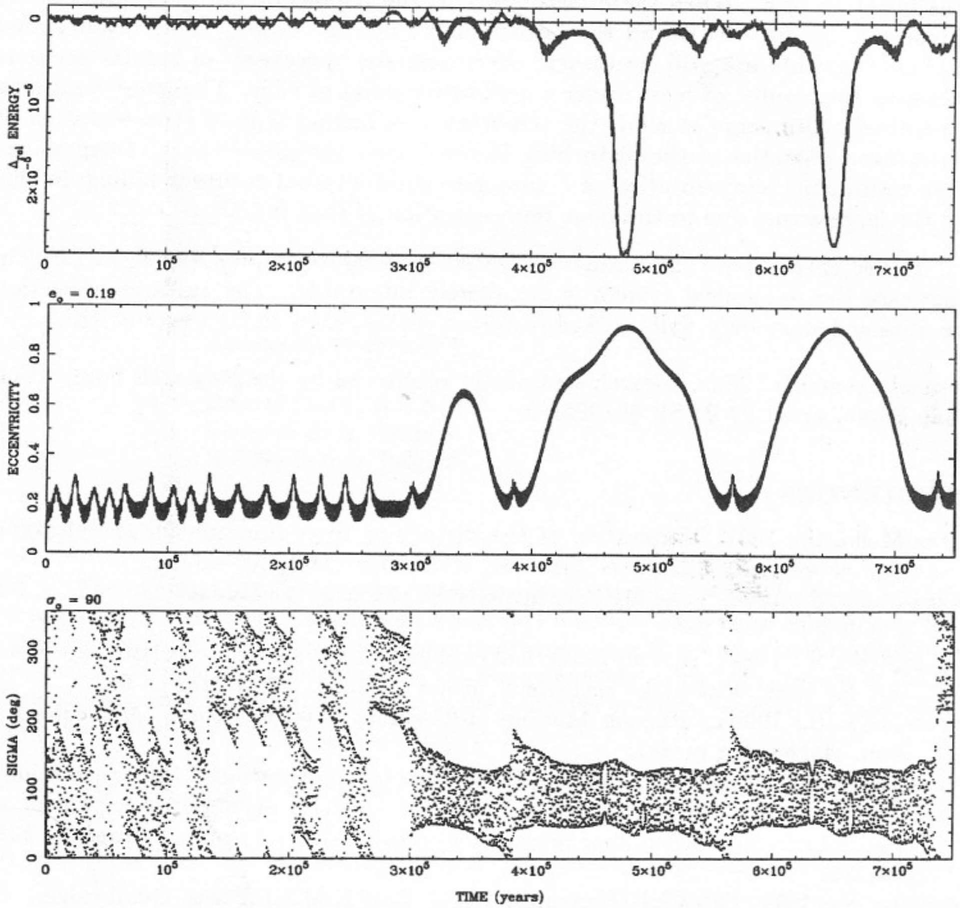

Fig. 7 – Variations of the energy (in units of E_S), the eccentricity and the critical angle σ, for 750,000 years, in the solution whose section is shown in fig. 5.

time long enough to allow him to be strongly perturbed by the Earth or Mars and to become, perhaps, a permanent Apollo.

It is worth mentioning that the asteroid (887) *Alinda* ($\Delta E = -6.6 \times 10^{-5} E_S$) is moving in the chaotic zone and may pass from one region to the other, undergoing variations of the eccentricity from 0.25 to 0.75 in 6×10^4 years. The secular evolution of its eccentricity and perihelion for 3×10^5 years is shown in figure 6 (the numerical integration of the exact equations of the motion shows variations in the range 0.35 – 0.76 since the capture of this asteroid in the resonance 3:1; see Milani *et al.* 1969). We emphasize the extreme sensitivity of this solution with respect to initial conditions. Very small changes are sufficient to give a crescent-like curve not including the origin.

The integrations were checked mainly by looking at the energy E. Inside a cell, the precision obtained is very good – ten figures, or more, for instance – but in the jump from one cell to the next, the results are impaired by the difference of the computed values of the function F in the two sides of the border. Figure 7 shows the behaviour of the energy of the solution shown in fig. 5, over 750,000 years. This figure also shows the evolution of the eccentricity and the angle σ, in the same time. The variation in the energy is clearly related to large eccentricities and large amplitudes of the libration of σ. When these factors occur, the regions of faster variation of A_0 are reached (see fig. 1(a)) and the border errors become larger. This result shows that improvements are still necessary. However, the succession of similar sections guarantees the results, at least under a qualitative point of view. The large variations in the energy also serve to show the importance of having a good representation of the averaged potential of the disturbing forces $-F$ over the phase space. Integrations with a continuous representation of F may give good internal accuracy hiding, in this way, the large errors due to the poor representation of that function.

Other checks were done making the eccentricity of Jupiter equal to zero, in which case the dynamical system is completely integrable. The surfaces of section thus obtained show only well marked-invariant curves, even at high eccentricities.

Acknowledgements. This research was partly sponsored by the Research Foundation of São Paulo, grant FAPESP 89/0964-9.

REFERENCES

Ferraz-Mello, S.: 1987, "Expansion of the disturbing force-function for the study of high-eccentricity librations", *Astron. Astrophys.* **183**, 397-402.

Ferraz-Mello, S.: 1988, "The high-eccentricity libration of the Hildas", *Astron. J.* **96**, 400-408.

Ferraz-Mello, S.: 1989, "A semi-numerical expansion of the averaged disturbing function for some very-high-eccentricity orbits", *Celest. Mech.* **45**, 65-68.

Ferraz-Mello, S.: 1990a, "Regular Motions of Resonant Asteroids", *Rev. Mexicana Astron. Astrof.* (in press).

Ferraz-Mello, S.: 1990b, "A theory of planar planetary corotations" (in preparation).

Ferraz-Mello, S., Sato, M.: 1989, "A very-high-eccentricity asymmetric expansion of the disturbing function near resonances of any order", *Astron. Astrophys.* **225**, 541-547.

Hagihara, Y.: 1971, *Celestial Mechanics*, Vol.II, Part I, M.I.T.Press, Cambridge.

Milani, A., Carpino, M., Hahn, G., Nobili, A.M.: 1989, "Project Spaceguard: Dynamics of planet- crossing asteroids. Classes of Orbital Behaviour", *Icarus* **78**, 212-269.

Morbidelli, A., Giorgilli, A.: 1990, "On the Dynamics of the Asteroidal Belt", *Celest. Mech. Dyn. Astron.* **47**, 145-204.

Silva, G.: 1909, "Sur les limites de convergence du développement de la fonction perturbatrice", *Bulletin Astron.* **26**, 49-75, 97-114.

Wisdom, J.: 1983, "Chaotic behaviour and the origin of the 3/1 Kirkwood gap", *Icarus* **56**, 51-74.

Wisdom, J.: 1985, "A perturbative treatment of motion near the 3/1 commensurability", *Icarus* **63**, 272-289.