Verification of Equation B.48

This short report deals with the analytical verification of the expression (B.48) of the book Canonical Perturbation Theories, Degenerate Systems and Resonance.

We know that

$$S = 4m\sqrt{k}\left[\mathcal{E}\left(\zeta, \frac{1}{\kappa}\right) + \beta\mathcal{F}\left(\zeta, \frac{1}{\kappa}\right)\right], \tag{1}$$

where

$$\kappa = \sqrt{\frac{2mk}{E+mk}}, \tag{2}$$

$$\beta = \frac{1-\kappa^2}{\kappa^2} = \frac{E-mk}{2mk}, \tag{3}$$

and the functions $\mathcal{F}$ and $\mathcal{E}$ in Eq. 1 correspond to the incomplete elliptic integrals of first and second kind, respectively.

The above relations are exactly the expressions given in the book, in the equations (B.45), (B.10), (B.47) and (B.20), respectively. It is worth mentioning that since we are in the libration regime of motion, $\kappa > 1$, and consequently $1/\kappa < 1$.

In order to obtain $\partial S/\partial E$, we have

$$\frac{\partial S}{\partial E} = 4m\sqrt{k}\left[\frac{\partial\mathcal{E}}{\partial\zeta}\frac{\partial\zeta}{\partial E} + \frac{\partial\mathcal{E}}{\partial(1/\kappa)}\frac{\partial(1/\kappa)}{\partial E} + \frac{\partial\beta}{\partial E}\mathcal{F} + \beta\frac{\partial\mathcal{F}}{\partial\zeta}\frac{\partial\zeta}{\partial E} + \beta\frac{\partial\mathcal{F}}{\partial(1/\kappa)}\frac{\partial(1/\kappa)}{\partial E}\right]. \tag{4}$$

The separate terms of each derivative are

$$\frac{\partial\beta}{\partial E} = \frac{1}{2mk},$$

$$\frac{\partial(1/\kappa)}{\partial E} = \frac{1}{4mk}\sqrt{\frac{2mk}{E+mk}} = \frac{\kappa}{4mk},$$

$$\frac{\partial\zeta}{\partial E} = \frac{\partial\zeta}{\partial\kappa}\frac{\partial\kappa}{\partial E} = -\frac{1}{2}\frac{\tan\zeta}{E+mk}.$$

The expressions for the derivatives of the elliptic integrals with respect to the arguments ζ and $1/\kappa$ can be found in expressions (710.07), (710.09), (730.00) and (730.01) of the Handbook of Elliptic Integrals for Engineers and Scientists. We have

$$\frac{\partial\mathcal{E}}{\partial\zeta} = \sqrt{1 - \frac{1}{\kappa^2}\sin^2\zeta},$$

$$\frac{\partial\mathcal{F}}{\partial\zeta} = \frac{1}{\sqrt{1-\frac{1}{\kappa^2}\sin^2\zeta}},$$

$$\frac{\partial\mathcal{E}}{\partial(1/\kappa)} = \kappa(\mathcal{E}-\mathcal{F}),$$

$$\frac{\partial\mathcal{F}}{\partial(1/\kappa)} = \frac{\kappa}{1-(1/\kappa^2)}\left[\mathcal{E} - \left(1-\frac{1}{\kappa^2}\right)\mathcal{F}\right] - \frac{1}{\kappa}\left(\frac{1}{1-(1/\kappa^2)}\right)\frac{\sin\zeta\cos\zeta}{\sqrt{1-(1/\kappa^2)\sin^2\zeta}}.$$

Calculating in separate parts the terms in Eq. 4, we have that

$$\frac{\partial\zeta}{\partial E}\left[\frac{\partial\mathcal{E}}{\partial\zeta}+\beta\frac{\partial\mathcal{F}}{\partial\zeta}\right]=-\frac{1}{2}\frac{\tan\zeta}{E+mk}\left[\sqrt{1-\frac{1}{\kappa^2}\sin^2\zeta}+\frac{E-mk}{2mk}\frac{1}{\sqrt{1-\frac{1}{\kappa^2}\sin^2\zeta}}\right],$$

$$=-\frac{1}{2}\frac{\tan\zeta}{E+mk}\left(1-\frac{1}{\kappa^2}\sin^2\zeta+\frac{E-mk}{2mk}\right)\frac{1}{\sqrt{1-\frac{1}{\kappa^2}\sin^2\zeta}},$$

and making one last simplification we get

$$\frac{\partial\zeta}{\partial E}\left[\frac{\partial\mathcal{E}}{\partial\zeta}+\beta\frac{\partial\mathcal{F}}{\partial\zeta}\right]=-\frac{1}{2}\frac{\tan\zeta}{E+mk}\frac{\cos^2\zeta}{\kappa^2}\frac{1}{\sqrt{1-\frac{1}{\kappa^2}\sin^2\zeta}}=-\frac{1}{4mk}\frac{\sin\zeta\cos\zeta}{\sqrt{1-\frac{1}{\kappa^2}\sin^2\zeta}}. \tag{5}$$

For the other key terms in Eq. 4, we have

$$\mathcal{F}\frac{\partial\beta}{\partial E}=\frac{1}{2mk}\mathcal{F}, \tag{6}$$

$$\frac{\partial(1/\kappa)}{\partial E}\left(\beta\frac{\partial\mathcal{F}}{\partial(1/\kappa)}+\frac{\partial\mathcal{E}}{\partial(1/\kappa)}\right)=\frac{1}{4mk}\left[-\mathcal{F}+\frac{\sin\zeta\cos\zeta}{\sqrt{1-\frac{1}{\kappa^2}\sin^2\zeta}}\right]. \tag{7}$$

Putting all the expressions together in the final expression for $\partial S/\partial E$, we obtain

$$\frac{\partial S}{\partial E}=4m\sqrt{k}\left[\frac{1}{2mk}\mathcal{F}-\frac{1}{4mk}\mathcal{F}-\frac{1}{4mk}\frac{\sin\zeta\cos\zeta}{\sqrt{1-\frac{1}{\kappa^2}\sin^2\zeta}}+\frac{1}{4mk}\frac{\sin\zeta\cos\zeta}{\sqrt{1-\frac{1}{\kappa^2}\sin^2\zeta}}\right], \tag{8}$$

and

$$\frac{\partial S}{\partial E}=\frac{1}{\sqrt{k}}\mathcal{F}\left(\zeta,\frac{1}{\kappa}\right). \tag{9}$$

The Eq. 9 is exactly the expression $(B.48)$ presented in the book.

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