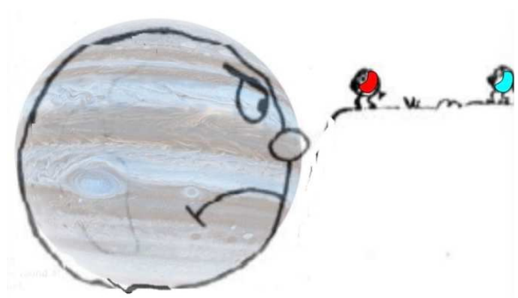
Sylvio Ferraz-Mello

DYNAMICS OF THE GALILEAN SATELLITES

An introductory treatise



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Preface (1979)

The launching of space probes to the outer Solar System started in the 1970 decade and the planning for Jupiter orbiter missions in the 1980 decade created a wide range of interest towards the motion of the Galilean satellites. In fact, current Jupiter mission planning includes extensive use of the Galilean satellites for dynamic orbit shaping to enhance scientific observation and mission performance.

Modern theoretical efforts involve both the revitalization of earlier theories and the proposal of completely new ones. In both cases, to obtain high precision, the theories deal particularly with the Galilean satellites and the understanding of the results requires a thorough study of their mathematical formulations. Nevertheless, a theory which uses the basic Lagrange's equations, is sufficient to explain and understand a lot of recent investigations on the motion of the Galilean satellites. The purpose of writing this book is to develop this suggestion. The theory is reconstructed following a classical presentation written by Tisserand in 1880 and the early theory of Laplace. Most of the materials presented in this book are based on the courses that I gave since 1972 to graduate students at the Aeronautics Institute of Technology and at the University of Sao Paulo. Little previous knowledge is expected of the reader.

Galilean satellites are very appropriate to introduce problems in Celestial Mechanics. If we disregard the effects of the Sun, of the oblateness of Jupiter and the libration, the Galilean satellites form a well-behaved *planetary system.* When the action of the Sun over one satellite is considered we have the *lunar problem.* When the oblateness of the planet is considered the satellites show the main features of the motion of *artificial satellites* while Jupiter displays *free nutation, precession* and *nutation.* At last, there is the *libration* produced by the resonance between the mean motions of the three inner satellites.

It is worthwhile to mention that the objective of the present book is neither to give all inequalities important to compute ephemerides nor to give accurate values for the main inequalities. For complete and precise results the reader is referred to the publications indicated at the end of the Chapters.

1 am indebted to Dr. P. D. Singh, Mr. M. Tsuchida and Miss S. M. Marcolino for help in the preparation of this book. The publication was supported by CNPq-Brazilian Council for Scientific and Technological Development.

Preface to this re-publication (2022)

Because of the discovery of several extra-solar planetary systems with planets forming resonant chains similar to the resonances between the Galilean satellites of Jupiter, the study of Laplace resonance is again among the top subjects of Celestial Mechanics. Laplace's theory of the motion of the satellites is thoroughly described in this book in several chapters culminating with one specific on the libration. However, the original version of this book (1979) was produced from typewritten camera-ready forms. The available scanned copies are yet impaired by a large number of corrections introduced by hand (1981). Given the rising interest in the subject and in order to allow a more comfortable reading, we decided to republish the book using new files produced with a LaTeX editor. This new edition is an unabridged reproduction of the revised 1981 version found in several repositories, with additionally the correction of some omissions and remaining misprints found when preparing the Russian edition (1983). This new edition is a contribution of the Brazilian section of the PLATO mission team and was done under the auspices of FAPESP (Proc.2016/13750-6) and CNPq (Proc. 303540/2020-6).

This book is dedicated to late Professor Carlos Alberto Buarque Borges in recognition of his open mind and spirit. He promoted the creation of the first graduate course in Astronomy in Brazil and provided support for a lot of academic work and research.

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Introduction

1.1 The Discovery

On the seventh day of January 1610, at one o'clock in the morning, when I was exploring the heavens with the telescope, Jupiter presented itself before my eyes, and because I had built an instrument of high precision I saw three small Stars close to it. Though I believed them to be Stars, they were ceaseless astonishing me because they seemed to lie exactly over one straight line parallel to the ecliptic and they were more splendid than other Stars of the same magnitude. Their positions were the following

that is, there were two on the eastern side and one on the west. The easternmost and the western one seemed to be slightly greater than the third one. I paid no attention to their distances from Jupiter for, as I have already told, I believed them to be fixed Stars. When, on the eight, led by what, I do not know, I returned to the same observation, I saw a completely different arrangement: the three Stars were now all in the western side, and they were closer to each other than in the day before, and at equal intervals from one another, as shown in the following drawing

Facing this phenomenon and unable to conceive that Stars could change relative positions, I began to hesitate and wonder how Jupiter could be east of these Stars when it had been west of two of them the day before. Would not its *motion be direct, at variance with the astronomical calculations, and would it by its own motion went beyond these Stars?*

I waited for the next night with impatience but I was disappointed in my hopes for the sky was cloudy everywhere.

On the tenth, however, the Stars appeared in the following positions with respect to Jupiter

They were just two Stars and both in the eastern side of Jupiter; the third was, I assumed, hidden behind the planet. They were, as before, in the same straight line with the planet and precisely along the Zodiac. Facing this fact and having understood that such mutations could not be attributed to Jupiter; yet convinced that the Stars were still the same, my hesitation was transformed into amazement. I understood that the apparent changes belonged not to Jupiter but to the Stars. For that reason, I decided to continue the observations with greater care and attention.

On the eleventh of the month, I saw this arrangement;

just two Stars east of Jupiter, the central one being three times as distant from Jupiter as from the other Star; the easternmost was two times greater than the central one whereas in the night before they appeared equal. I admitted since them that there existed in the heavens, without any doubt, Stars turning around Jupiter in the same way in which Mercury and Venus turn around the Sun...

In these words, Galileo told the discovery of the four great satellites of Jupiter. The deep significance of this discovery has not been paralleled many times in the history of Astronomy. It would be nowadays comparable, for example, to a discovery of life on Mars or the detection by radioastronomers of signals arising from some extra-terrestrial civilization. The discovery of the four satellites served to remove what was a great objection to the motion of the Earth. The objection was that though all the planets turn around the Sun, the Earth alone is not solitary but goes together in the company of the Moon around the Sun in one year while at the same time the Moon moves around the Earth every month. The discovery of satellites of Jupiter removed this apparent anomaly of the theory of Copernicus, for Jupiter, like another Earth, goes around the Sun, in twelve years accompanied not by one but by four moons.

1.2 Jupiter's Satellites

Today, thirteen satellites of Jupiter are known. The last satellite discovered is Jupiter XIII (Leda) found by C.T.Kowal on plates taken on September 1974, with the 1.2-meter Schmidt telescope of the Palomar Observatory. The visual magnitude of Leda is approximately 20 and it is estimated to be less than 8 kilometers in diameter.

The main characteristics of the Satellites of Jupiter are shown in Table 1.1.

Table 1.1. Jupiter's Satellites

Number	Name	Semi-major Axis (10^5 km)	Eccen-tricity	Inclination (degrees)	Period (days)	Visual Magnitude
V	Amalthea	1.81	0.0028	0.5	0.50	13 .0
I	Io	4.22	0.0042*	0.04	1.77	5.0
II	Europa	6.71	0.0094*	0.47	3.55	5.3
III	Ganymede	10.7	variable*	0.19	7.16	4.6
IV	Callisto	18.8	0.0073	0.25	16.7	5.6
XIII	Leda	111	0.148	27.8	239	20
VI	Himalia	115	0.158	27.6	251	14.8
X	Lisythea	117	0.130	29.0	260	18.4
VII	Elara	117	0.207	24.8	260	16.4
XII	Ananke	207	0.169	147 (-33)	617	18.9
XI	Carme	224	0.207	164 (-16)	692	18.0
VIII	Pasiphaë	233	0.378	145 (-35)	735	17,7
IX	Sinope	237	0.275	153 (-27)	758	18.3

* See Section 6.4

For the Galilean satellites and Jupiter V (Amalthea), the inclination is referred to the equatorial plane of Jupiter. For the distant satellites, the orbital plane of the planet is more relevant. The osculating eccentricities of Jupiter I (Io), Jupiter II (Europa) and Jupiter III (Ganymede) are discussed in Section 6.4.

Satellites VI to XII were named in 1975 by the Working Group for Planetary System Nomenclature of the International Astronomical Union. The names follow the traditions established by the existing names in the system. The name Leda was proposed by the discoverer of Jupiter XIII. The outer satellites with direct orbits have names ending in a. The outer satellites with retrograde orbits have names ending in e.

If we classify the satellites by their physical and orbital parameters, we find that almost all of them belong to one of three groups:

(a) *Galilean satellites*. Massive satellites orbiting very close to the planet in very regular equatorial orbits.

(b) *Himalia group*. Small satellites with direct orbits at an average distance of about 11 million kilometers from the planet with very similar inclinations and eccentricities. The similarity of orbital elements has indicated to some that the members of this group result from the same event. Capture has been suggested but the apparent lack of large brightness variations for any satellite and the unusual color of Jupiter VI (Himalia) argue that these satellites may not be simply captured asteroids.

(c) *Pasiphaë group*. Very irregular group of satellites with retrograde orbits at an average distance of 22 million kilometers from the planet, eccentric, and inclined 18° to 35° over the orbital plane of the planet. Jupiter VIII (Pasiphaë) has the distinction of attaining a greater distance from its primary than other known satellites in the Solar System: 33 million kilometers. Jupiter IX (Sinope) completes its revolution in 2.07 years and has the longest period of revolution among the known planetary satellites. There is almost general agreement that all of them must have been captured, but so far there is no detailed theory which explains the capture of them. It seems reasonable that there is a connection between these satellites and the Trojans and it is possible that the satellites are captured asteroids; perhaps shortly following the Solar System's origin, when the space density of asteroids in the vicinity of Jupiter's orbit was considerably higher than it is today.

Jupiter V (Amalthea) is very small and too far away from the first group to be a member. It is the only observed member of another group of less massive Jovian satellites.

Analytical theories of the motion of outer satellites are difficult to derive since the eccentricities and the ratio of the mean motions of the satellite and Jupiter are large. The most current and efficient tool is numerical integration which gives an accuracy of a few arcseconds. Some noticed near commensurabilities of the satellites mean motions and the jovicentric mean motion of the Sun. In fact, the only noticeable near-commensurability from Table 1.1 happens for Jupiter XII (Ananke) $n_{12} - 7n_0 = 0.0018$ deg/day. This value is very sensitive to improvements in the period of the satellite and, after all, the given determination lays over no more than a dozen revolutions (Ananke was discovered in 1951).

A probable fourteenth satellite, of photovisual magnitude 21, was picked up by Kowal with the Schmidt telescope on September-October, 1975. Not enough observations were obtained to allow the determination of an orbit for this object, but a heliocentric orbit has been ruled out. It was hoped that more observations could be obtained at the following oppositions of Jupiter, but no additional observations have been reported. Presumably, this object will be rediscovered at some opposition in the future.

1.3 The Galilean Satellites

The Galilean satellites form with the Moon, Saturn VI (Titan) and Neptune I (Triton), a family of giant satellites with masses ranging from 5×10^{25}g (Europa) to 15×10^{25}g (Triton, Titan and Ganymede). The remaining satellites in the Solar System have masses at least 10 times smaller. Ganymede would be comparatively an easy naked-eye object, were it not for the proximity of the bright planet; the other Galileans would be near the limit. A very modest pair of binoculars will reveal them all.

The most interesting and easily observed phenomena of these bodies are their eclipses, their occultations and their transits across the disk of the planet. Also, when one satellite is in transit across the disk, the shadow it projects on the face of the planet can generally be seen.

The inner satellites pass through the shadow of Jupiter at superior conjunction, and across his disk at every inferior conjunction. Callisto is the only one that is far enough away from the planet ever to pass above or below the shadow and the disk when the conjunctions are distant of the line of nodes of the satellite orbit.

The distances of Ganymede and Callisto are large enough to allow both disappearance and reappearance at a single eclipse to be observed on the same side of the planet when the angle between the Earth and the Sun, as seen from Jupiter, is sufficiently large.

Twice each Jovian year (which is 11.86 Earth years long) the plane containing the satellite orbits passes through the Sun and for some three to six months both the Sun and the Earth remain close to that plane. Then, mutual occultations and mutual eclipses may happen. Mutual phenomena are very important since their observation provides us with the most precise data for the study of the motion of satellites.

The theory of the Galilean satellites is one of the most interesting in Celestial Mechanics. There is a conspicuous relation between the mean motion of the three inner Galilean satellites:

$$n_1 - 3n_2 + 2n_3 = 0$$

It was treated with profound skill by Laplace (see Chapter VII). Emendations to this theory were given by Souillart, Tisserand and Sampson. Modern improvements and results are due to Marsden and Brown. Laplace showed that if the mean longitudes and mean motions are such that the angle $\lambda_1 - 3\lambda_2 + 2\lambda_3$ differed a little from 180°, there was a minute restoring force arising from the mutual actions of the satellites, tending to bring this angle toward the value 180°. Thus, oscillations will be produced in virtue of which the angle will oscillate very slowly on each side of the central value. This is the phenomenon called *Libration* of the Galilean satellites.

Sampson's tables of the four great satellites, published in 1910, have for many years been the only available source for the prediction of the phenomena and of satellite positions. They are now 70 years old. New theories are now under study in Brazil, France and United States. Sampson's theory and tables have been rejuvenated by J. Lieske at the Jet Propulsion Laboratory; Lieske's subroutines package for the computation of ephemeris is the best available today.

Current Jupiter orbiter mission planning includes extensive use of gravitational fields of the Galilean satellites for dynamic orbit shaping to enhance scientific observation and mission performance. During the course of the nominal operational lifetime of such orbiters (l-2 years), some degree of active control based on real-time adaptive orbit design must be in effect so to avoid premature collision with the satellites. Accuracy requirements in the positions of the Galilean satellites are 400 km in the case of Voyager 1 and Voyager 2 missions. In these missions, the probes are catapulted by the powerful Jovian gravity toward Saturn, Uranus and even Neptune, after close-up photographic surveys of Jupiter. Accuracy requirements for the Jupiter Orbiter Probe (*Galileo*) are still tighter (50-100 km).

The evolution of the system is an open question. Some claim that the observed resonances among the mean motions are due to dissipative forces like drag, efficient in the early stages of the formation of the Solar System, and tides. Some, however, following ideas first exposed by Roy and Ovenden on the occurrence of commensurable mean motions in the Solar System, argue that conservative evolution is sufficient to explain the present situation. Recent calculations show that the time of gravitational evolution necessary to get the system close to the present configuration, in which the time-mean of the action associated with the mutual interaction of the satellites is a minimum, is closely comparable with the age of the Solar System.

References and Notes

- 1.1
  The description of the discovery of the satellites is a translation of some parts of Galileos's *Siderius Nuncius* (The Starry Messenger) published in Venice in 1610.

- 1.2
  Data in Table 1.1 are mostly from
  J.A.Burns (ed.): 1977, *Planetary Satelites*. Univ. Arizona Press, Tucson.
  Data relative to the Galilean satellites are those discussed in this book. See also
  H.Alfvén and G.Arrhenius: 1976, *Evolution of the Solar System*. NASA SP-345, Washington.

- 1.3

The classical theory used all along this century is

R.A.Sampson: 1921, "Theory of the Four Great Satellites of Jupiter", *Memoirs Royal Astron. Soc.* Vol. LXIII.

and its revitalization

J.Lieske: 1977, "Theory of Motion of Jupiter's Galilean Satellites". *Astron. Astrophys.* 56, 333-352.

The construction of new theories, in progress, is founded on

S.Ferraz-Mello, 1966, "Recherches sur le Mouvement des Satellites Galiléens de Jupiter", *Bull. Astronomique*, série 3, vol.1, 287-330.

J.L.Sagnier: 1973, "Contribution à l'étude dynamique du Système Galiléen de Jupiter", *Astron. Astrophys* 25, 113-124.

S.Ferraz-Mello: 1974, "On the Theory of the Galilean Satellites of Jupiter". In Y. Kozai (ed.), *The Stability of the Solar System and of Small Stelar Systems*, D. Reidel, Dordrecht, pp. 167-184.

S.Ferraz-Mello: 1978, "A Second-order Theory of the Galilean Satellites of Jupiter". In V.Szebehely (ed.), *Dynamics of Planets and Satellites and Theories of their Motion*, D. Reidel, Dordrecht, pp. 209-236.

J.L.Sagnier: 1981, *Le Mouvement des Satellites Galiléens de Jupiter* Thèse de Doctorat, Univ. Pierre et Marie Curie, Paris.

Theories using other techniques are

W.de Sitter: 1925, "New Mathematical Theory of Jupiter's Satellites" *Annalen Sterrewacht Leiden*, vol. XII.

B.Marsden: 1966, *The Motions of the Galilean Satellites of Jupiter*, Ph.D. Dissertation, Yale University, New Haven.

A complete study of the free oscillations and long-period inequalities is

B.C. Brown: 1977, "The Long Period Behavior of the Orbits of the Galilean Satellites of Jupiter", *Celestial Mechanics*, 16, 229-259.

Brown's numerical results are extensively used in Chapters V to X for sake of comparison.

2

The Equations

2.1 Mutual Interactions

Before deriving the equations of variation of the Keplerian elements, we shall describe the equations of the motion of a system of satellites in its more general vector form. Let the central body and surrounding satellites be considered as a set of $n+1$ point masses $m_0, m_1, \cdots, m_n$ at the positions (vectors) $\boldsymbol{r}_0, \boldsymbol{r}_1, \cdots, \boldsymbol{r}_n$ and the distance between the two masses, m_i and m_j, given by r_{ij}. The force arising from m_j and acting on m_i is

$$\boldsymbol{f}_{ij} = Gm_im_j \frac{\boldsymbol{r}_j - \boldsymbol{r}_i}{r_{ij}^3}; \tag{2.1}$$

the value of G depends on the chosen units of mass, time and distance ($6670 \pm 5 \times 10^{-11}$ cgs units). The force $\boldsymbol{f}_{ij}$ is radial and its absolute value depends only on the distance r_{ij} and on extrinsic physical quantities (the masses), In this case, we have $\mathrm{curl}\boldsymbol{f}_{ij} = 0$ and the force $\boldsymbol{f}_{ij}$ arises from a force field whose potential is

$$\omega_{ij} = \int \boldsymbol{f}_{ij} \cdot d(\boldsymbol{r}_i - \boldsymbol{r}_j)$$

i.e.

$$\omega_{ij} = -Gm_im_j \int \frac{(\boldsymbol{r}_i - \boldsymbol{r}_j) \cdot d(\boldsymbol{r}_i - \boldsymbol{r}_j)}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3}.$$

The above integral is equal to $-|\boldsymbol{r}_i - \boldsymbol{r}_j|^{-1}$ and then

$$\omega_{ij} = \frac{Gm_im_j}{r_{ij}} + \text{const.}$$

The integration constant is determined by the normalizing condition $\lim\limits_{r_{ij}\to\infty} \omega_{ij} = 0$, which makes the integration constant equal to zero. Thus

$$\omega_{ij} = \frac{Gm_im_j}{r_{ij}}$$

and

$$\boldsymbol{f}_{ij} = \mathrm{grad}_{\boldsymbol{r}_i} \omega_{ij}.$$

The operator $\mathrm{grad}_{\boldsymbol{r}_i}$ has a very precise meaning; it is not intrinsic since it depends upon the origin which shall be precisely defined. In this case, the origin is at $\boldsymbol{r}_j$. Then

$$\mathrm{grad}_{\boldsymbol{r}_i} = \boldsymbol{\nabla}_i r_{ij} \frac{\partial}{\partial r_{ij}}$$

where

$$\boldsymbol{\nabla}_i r_{ij} = \frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{r_{ij}}$$

(unit vector). Introducing a new operator

$$\boldsymbol{\nabla}_{ij} = \boldsymbol{\nabla}_i r_{ij} \frac{\partial}{\partial r_{ij}},$$

we get

$$\boldsymbol{f}_{ij} = \boldsymbol{\nabla}_{ij}\ \omega_{ij}. \tag{2.2}$$

2.2 Equations of Motion

The total force acting on m_i is

$$\boldsymbol{f}_i = \sum\nolimits^* \boldsymbol{f}_{ij} = \sum\nolimits^* G m_i m_j \frac{\boldsymbol{r}_j - \boldsymbol{r}_i}{r_{ij}^3}; \tag{2.3}$$

where $\sum^*$ represents a sum over the subscript j for all j from 0 to n excepted $j = i$. Using distributive properties of ∇-operators, we get $\mathrm{curl}\boldsymbol{f}_i = 0$, which shows that the field in $\boldsymbol{r}_i$ arising from the superposition of the individual fields is also potential. If

$$\Omega^* = \sum_{i=0}^{n} \sum_{j>i} \omega_{ij},$$

equations (2.2) and (2.3) can be written as, respectively,

$$\boldsymbol{f}_{ij} = \boldsymbol{\nabla}_{ij} \Omega^*$$

and

$$\boldsymbol{f}_i = \sum_{j=0}^{n} \boldsymbol{\nabla}_{ij} \Omega^*$$

If the system of reference is an inertial Galilean frame, the Newton's laws of motion are applicable and the equations of the motion are

$$m_i \ddot{\boldsymbol{r}}_i = \sum_{j=0}^{n} \boldsymbol{\nabla}_{ij} \Omega^* \tag{2.4}$$

These equations form a differential system of $6(n+1)^{th}$ order possessing ten known first integrals.

2.3 Planetocentric Equations

In the case of a system of satellites (or a system of planets) whose individual masses are $m_1, m_2, \cdots, m_n$, orbiting around a central massive body of mass m_0 where $m_i \ll m_0$ $(i = 1, \cdots, n)$, it is a wise step to introduce a new reference frame whose origin is kept fixed in the central mass and the axes are kept parallel to those of the Galilean frame. In this Copernican frame, the accelerations are

$$(\boldsymbol{r}_i - \boldsymbol{r}_0)^{\cdot\cdot} = \ddot{\boldsymbol{r}}_i - \ddot{\boldsymbol{r}}_0$$

and the equations of the motion are

$$m_i(\boldsymbol{r}_i - \boldsymbol{r}_0)^{\cdot\cdot} = \sum_{j=0}^{n} \left(\boldsymbol{\nabla}_{ij} - \frac{m_i}{m_0} \boldsymbol{\nabla}_{0j} \right) \Omega^* \tag{2.5}$$

These equations form a differential system of the $6n^{th}$ order. If the solutions of equation (2.5) are known, the law of conservation of momentum and the reduction to the centre of mass allow us to have the solution of the equations (2.4). In astronomy, this step is often unnecessary since in general only relative motions are considered. The Copernican equations possess four first integrals, which are not as simple as in the Galilean frame, and, in practice, their use does not lead to a simpler problem. Also, if they are used, the symmetrical and simple shape of equations (2.5) disappear. The only exception is the case of three bodies for which Lagrange succeeded to maintain the symmetrical shape of the equations by means of a set of very subtle transformations.

Introducing Ω_0 and Ω defined by

$$\Omega_0 = \sum_{j=1}^{n} \omega_{0j} \qquad\qquad \Omega = \sum_{i=1}^{n} \sum_{j>i} \omega_{ij}$$

we have

$$\begin{aligned} \boldsymbol{\nabla}_{ij}\Omega^* &= \boldsymbol{\nabla}_{ij}\Omega_0 + \boldsymbol{\nabla}_{ij}\Omega \\ \boldsymbol{\nabla}_{0j}\Omega^* &= -\boldsymbol{\nabla}_{j0}\Omega_0 \end{aligned}$$

and the equations of motion of the satellites in the Copernican planetocentric frame are

$$m_i(\boldsymbol{r}_i - \boldsymbol{r}_0)^{\cdot\cdot} = \left(1 + \frac{m_i}{m_0}\right) \boldsymbol{\nabla}_{i0}\Omega_0 + \sum_{j \neq i} \left(\boldsymbol{\nabla}_{ij}\Omega + \frac{m_i}{m_0} \boldsymbol{\nabla}_{j0}\Omega_0 \right) \tag{2.6}$$

The forces in the right-hand side are well known. We have the central Keplerian attraction

$$\left(1 + \frac{m_i}{m_0}\right) \boldsymbol{\nabla}_{i0}\Omega_0$$

and the disturbing force

$$\sum_{j \neq i} \left(\boldsymbol{\nabla}_{ij} \mathit{\Omega} + \frac{m_i}{m_0} \boldsymbol{\nabla}_{j0} \mathit{\Omega}_0 \right). \tag{2.7}$$

In the disturbing force, the first term, referred as *direct disturbing force*, accounts for the direct action of all other satellites over the i^{th} satellite; the second term, referred as *indirect disturbing force*, accounts for the counterpart over the i^{th} satellite, of the action of the other satellites on the motion of the central body.

2.4 Rotating Frame

Let us consider a new frame which is rotating with a constant angular velocity with respect to the Copernican frame. The Coriolis theorem writes

$$\boldsymbol{a}_i + 2\boldsymbol{\omega} \times \boldsymbol{v}_i = (\boldsymbol{r}_i - \boldsymbol{r}_0)^{\cdot\cdot} - \boldsymbol{\omega} \times \big(\boldsymbol{\omega} \times (\boldsymbol{r}_i - \boldsymbol{r}_0)\big).$$

We may notice that curl $\big[\boldsymbol{\omega} \times \big(\boldsymbol{\omega} \times (\boldsymbol{r}_i - \boldsymbol{r}_0)\big)\big] = 0$. Indeed, because of the rules of the vector triple product,

$$\mathrm{curl}\big[\boldsymbol{\omega} \times \big(\boldsymbol{\omega} \times (\boldsymbol{r}_i - \boldsymbol{r}_0)\big)\big] = \boldsymbol{\omega} \cdot \mathrm{div}\big(\boldsymbol{\omega} \times (\boldsymbol{r}_i - \boldsymbol{r}_0)\big) - \big(\boldsymbol{\omega} \times (\boldsymbol{r}_i - \boldsymbol{r}_0)\big) \cdot \mathrm{div}\ \boldsymbol{\omega}.$$

Since $\boldsymbol{\omega}$ is constant, the last term on the right hand side of this equation is zero, and using the invariance property of the scalar triple product, we have

$$\mathrm{div}\big(\boldsymbol{\omega} \times (\boldsymbol{r}_i - \boldsymbol{r}_0)\big) = (\boldsymbol{r}_i - \boldsymbol{r}_0) \cdot \mathrm{curl}\ \boldsymbol{\omega} = 0.$$

Thus, the curl of the centrifugal acceleration of the i^{th} satellite is zero. The centrifugal acceleration arises from the potential

$$\sigma_i = -\int \big[\boldsymbol{\omega} \times \big(\boldsymbol{\omega} \times (\boldsymbol{r}_i - \boldsymbol{r}_0)\big)\big] \cdot d(\boldsymbol{r}_i - \boldsymbol{r}_0),$$

i.e.

$$\sigma_i = \frac{1}{2}\omega^2 |\boldsymbol{r}_i - \boldsymbol{r}_0|^2 - \frac{1}{2}\big(\boldsymbol{\omega} \cdot (\boldsymbol{r}_i - \boldsymbol{r}_0)\big)^2;$$

the integration constant is zero, which means that the centrifugal potential is normalized and is zero at the origin. We can still write:

$$\sigma_i = \frac{1}{2}\big(\boldsymbol{\omega} \times (\boldsymbol{r}_i - \boldsymbol{r}_0)\big)^2$$

and the equations of the motion of the i^{th} satellite in the new frame are

$$\boldsymbol{a}_i + 2\boldsymbol{\omega} \times \boldsymbol{v}_i = \boldsymbol{\nabla}_{i0}\sigma_i + \frac{m_0 + m_i}{m_0 m_i} \boldsymbol{\nabla}_{i0}\sigma_0 + \sum_{j \neq i} \left(\boldsymbol{\nabla}_{ij} \frac{\mathit{\Omega}}{m_i} + \boldsymbol{\nabla}_{j0} \frac{\mathit{\Omega}_0}{m_0} \right).$$

In this equation, several ∇-operators have been used. To have a more homogeneous equation, the direct and indirect disturbing forces are to be modified.. By using the definitions of these operators, we have

$$\sum_{j\neq i} \boldsymbol{\nabla}_{j0} \frac{\Omega_0}{m_0} = -\boldsymbol{\nabla}_i \sum_{j\neq i} Gm_j \frac{\boldsymbol{r}_i \cdot \boldsymbol{r}_j}{r_{j0}^3}$$

and

$$\sum_{j\neq i} \boldsymbol{\nabla}_{ij} \frac{\Omega}{m_i} = \boldsymbol{\nabla}_i \frac{\Omega}{m_i}.$$

The equations of the motion become

$$\boldsymbol{a}_i + 2\boldsymbol{\omega} \times \boldsymbol{v}_i = \boldsymbol{\nabla}_i \left(\sigma_i + \frac{m_0 + m_i}{m_0 m_i} \Omega_0 + \frac{\Omega}{m_i} - \sum_{j\neq i} Gm_j \frac{\boldsymbol{r}_i \cdot \boldsymbol{r}_j}{r_{j0}^3} \right)$$

or

$$\boldsymbol{a}_i + 2\boldsymbol{\omega} \times \boldsymbol{v}_i = \boldsymbol{\nabla}_i \left(\frac{1}{2} (\boldsymbol{\omega} \times \boldsymbol{r}_i)^2 + \frac{G(m_0 + m_i)}{r_{i0}} + \sum_{j\neq i} Gm_j (\frac{1}{r_{ij}} - \frac{\boldsymbol{r}_i \cdot \boldsymbol{r}_j}{r_{j0}^3}) \right). \tag{2.8}$$

2.5 Equations in Rotating Coordinates

Depending on the kind of study, the rotating frame should be emphasized. We may take $\boldsymbol{\omega}$ perpendicular to the fundamental plane of the reference system and positively oriented.

In rectangular coordinates, the ∇-operator is

$$\boldsymbol{\nabla}_i = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_i} \right);$$

also $\boldsymbol{\omega} = (0, 0, N)$ and the Eulerian equations of the motion are

$$\begin{aligned} \ddot{x}_i - 2N\dot{y}_i &= \frac{\partial W_i}{\partial x_i} \\ \ddot{y}_i + 2N\dot{x}_i &= \frac{\partial W_i}{\partial y_i} \\ \ddot{z}_i &= \frac{\partial W_i}{\partial z_i} \end{aligned}$$

where

$$W_i = \frac{1}{2} N^2 (x_i^2 + y_i^2) + G\frac{m_0 + m_i}{r_{i0}} + \sum_{j\neq i} Gm_j \left(\frac{1}{r_{ij}} - \frac{\boldsymbol{r}_i \cdot \boldsymbol{r}_j}{r_{j0}^3} \right).$$

Similarly, in cylindrical coordinates, the Eulerian equations of the motion are

$$\begin{aligned}
\ddot{\rho}_i - \rho_i \dot{\phi}_i^2 - 2N\rho_i\dot{\phi}_i &= \frac{\partial W_i}{\partial \rho_i} \\
\rho_i\ddot{\phi}_i + 2\dot{\rho}_i\dot{\phi}_i - 2N\dot{\rho}_i &= \frac{1}{\rho_i}\frac{\partial W_i}{\partial \phi_i} \\
\ddot{z}_i &= \frac{\partial W_i}{\partial z_i}.
\end{aligned}$$

2.6 Application to the Galilean Satellites

All these ways of describing the motion of a system of satellites have been used on several occasions. In modern theoretical studies of the Galilean satellites, the equations referred to the Eulerian frame have been preferred. The choice of the rotation velocity of the frame is made on the grounds of one special feature of the problem:

$$\boxed{\begin{array}{c}\text{The mean motions of the three} \\ \text{inner satellites are such that} \\ n_1 - 2\,n_2 = n_2 - 2\,n_3\end{array}} \tag{2.9}$$

and the rotation is chosen in such a way that the mean motions referred to the Eulerian frame

$$\nu_i = n_i - N$$

are such that

$$4\nu_3 = 2\nu_2 = \nu_1$$

The Eulerian equations of motion have played important role in several classical studies of the motion of the Moon. Euler, in his pioneer work, referred the motion to a rotating frame whose rotation velocity was the sidereal mean motion of the Moon. G. W. Hill, in his celebrated work, considers the same equations, but the frame rotates following the mean motion of the Sun.

In the Laplacian theory of the Galilean satellites, the equations are Lagrange's equations of variation of the elements. We will derive these equations, starting from the planetocentric equations of the motion of the satellites.

2.7 Keplerian Elements

If we neglect the mutually disturbing action of the satellites, the forces acting on each satellite are reduced to $(1 + m_i/m_0)\boldsymbol{\nabla}_i \Omega_0$ and the resulting motion is described by an ellipse, the centre of the planet being at its one focus. This is a two-body problem and we assume that its solution is known:

$$\begin{aligned}
x &= r\big(\cos\Omega\cos(f+\omega) - \cos I\sin\Omega\sin(f+\omega)\big)\\
y &= r\big(\sin\Omega\cos(f+\omega) + \cos I\cos\Omega\sin(f+\omega)\big)\\
z &= r\sin I\sin(f+\omega)\\
r &= \frac{a(1-e^2)}{1+e\cos f}\\
f &= \ell + 2e\sin\ell + \frac{5}{4}e^2\sin 2\ell + \cdots\\
\ell &= nt + \sigma.
\end{aligned} \tag{2.10}$$

Through these equations the coordinates x, y, z of one satellite are related to the elements of its osculating orbit: the semi-major axis a, the eccentricity e, the inclination I, the longitude of the ascending node Ω, the argument of the perijove ω and the mean anomaly of the epoch σ. As auxiliary quantities are the true anomaly f, the mean anomaly ℓ and the mean motion n.

2.8 Variation of the Elements

We shall also consider the inverse problem, namely, the determination of the six elements of a satellite orbit when the position vector $\boldsymbol{r}$ and the corresponding velocity $\boldsymbol{v}$ are known. The calculation of the elements leads for every orbital element C_i to a relation

$$C_i = C_i(\boldsymbol{r}, \boldsymbol{v})$$

$(i = 1, \cdots, 6)$ and for its variation

$$\dot{C}_i = \dot{\boldsymbol{r}} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} C_i + \dot{\boldsymbol{v}} \cdot \boldsymbol{\nabla}_{\boldsymbol{v}} C_i$$

or

$$\dot{C}_i = \boldsymbol{v} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} C_i + \ddot{\boldsymbol{r}} \cdot \boldsymbol{\nabla}_{\boldsymbol{v}} C_i. \tag{2.11}$$

The subscripts $\boldsymbol{r}$ and $\boldsymbol{v}$ in the ∇-operators indicate whether the gradient is taken with respect to the coordinates or to the components of the velocity.

Like Eulerian equations, the planetocentric equations (2.6) can also be transformed and the result would be the same if $\boldsymbol{\omega}$ was made equal to zero in equations (2.8). If for sake of simplicity we write $\boldsymbol{r}_i$ instead of $\boldsymbol{r}_i - \boldsymbol{r}_0$, the equations for the planetocentric motion are

$$\ddot{\boldsymbol{r}} = \boldsymbol{\nabla}_i \left(G\frac{m_0 + m_i}{r_{i0}} + \sum_{j \neq i} G m_j \left(\frac{1}{r_{ij}} - \frac{\boldsymbol{r}_i \cdot \boldsymbol{r}_j}{r_{j0}^3} \right) \right). \tag{2.12}$$

In order to obtain the equations of variation of the elements of one satellite, let the p1anetocentric equation of its motion be written as

$$\ddot{\boldsymbol{r}} = \boldsymbol{\nabla}_{\boldsymbol{r}} F_0 + \boldsymbol{\nabla}_{\boldsymbol{r}} R$$

where F_0 represents the central Kep1erian attraction and R the disturbing force-function per unit mass. Equation (2.11) then becomes

$$\dot{C}_i = \boldsymbol{v} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} C_i + (\boldsymbol{\nabla}_{\boldsymbol{r}} F_0 + \boldsymbol{\nabla}_{\boldsymbol{r}} R) \cdot \boldsymbol{\nabla}_{\boldsymbol{v}} C_i.$$

In case of undisturbed motion of the satellite, the elements C_i do not vary ($\dot{C}_i = 0$), and the following relation must hold

$$0 = \boldsymbol{v} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} C_i + \boldsymbol{\nabla}_{\boldsymbol{r}} F_0 \cdot \boldsymbol{\nabla}_{\boldsymbol{v}} C_i.$$

The variation of the elements then results

$$\dot{C}_i = \boldsymbol{\nabla}_{\boldsymbol{r}} R \cdot \boldsymbol{\nabla}_{\boldsymbol{v}} C_i.$$

i.e.

$$\dot{C}_i = \sum_j \frac{\partial C_i}{\partial v_j} \frac{\partial R}{\partial x_j}.$$

Also

$$\sum_i \frac{\partial v_k}{\partial C_i} \dot{C}_i = \sum_j \sum_i \frac{\partial v_k}{\partial C_i} \frac{\partial C_i}{\partial v_j} \frac{\partial R}{\partial x_j} = \frac{\partial R}{\partial x_k}$$

and

$$\sum_k \sum_i \frac{\partial v_k}{\partial C_i} \frac{\partial x_k}{\partial C_j} \dot{C}_i = \sum_k \frac{\partial R}{\partial x_k} \frac{\partial x_k}{\partial C_j} = \frac{\partial R}{\partial C_j}.$$

In an analogous way, considering that R does not depend on the velocities, we have

$$\sum_k \sum_i \frac{\partial v_k}{\partial C_j} \frac{\partial x_k}{\partial C_i} \dot{C}_i = 0$$

and we obtain the symmetrical equations

$$\sum_i [C_j, C_i] \dot{C}_i = \frac{\partial R}{\partial C_j} \tag{2.13}$$

where $[C_j, C_i]$ are Lagrange's brackets defined by

$$[C_j, C_i] = \sum_k \left(\frac{\partial x_k}{\partial C_j} \frac{\partial v_k}{\partial C_i} - \frac{\partial x_k}{\partial C_i} \frac{\partial v_k}{\partial C_j} \right). \tag{2.14}$$

2.9 Invariance of the Brackets

The calculus of the brackets offers a great deal of simplification on account of the property

$$\frac{d}{dt}[C_j, C_i] = 0; \tag{2.15}$$

Indeed, we have

$$\frac{d}{dt}[C_j, C_i] = \sum_k \left(\frac{\partial x_k}{\partial C_j}\frac{\partial a_k}{\partial C_i} - \frac{\partial x_k}{\partial C_i}\frac{\partial a_k}{\partial C_j} \right)$$

where the radius vector $\boldsymbol{r}$ and the acceleration $\boldsymbol{a}$ are related to the osculating elements C_i through the solution of the problem of two bodies:

$$\boldsymbol{a} = \boldsymbol{\nabla}_{\boldsymbol{r}} F_0$$

so that

$$\frac{\partial a_k}{\partial C_i} = \sum_j \frac{\partial a_k}{\partial x_j}\frac{\partial x_j}{\partial C_i} = \sum_j \frac{\partial^2 F_0}{\partial x_j \partial x_k}\frac{\partial x_j}{\partial C_i},$$

and

$$\sum_k \frac{\partial x_k}{\partial C_j}\frac{\partial a_k}{\partial C_i} = \sum_k \sum_j \frac{\partial^2 F_0}{\partial x_l \partial x_k}\frac{\partial x_l}{\partial C_i}\frac{\partial x_k}{\partial C_j}.$$

The invariance of the brackets stated in equation (2.15) follows from the invariance of the above formulae to the exchange of the subscripts i and j.

2.10 Lagrange's Variational Equations

Lagrange's brackets may be calculated at a fixed point of the orbit and the periapsis offers the best point at which they should be considered. Computing all the derivatives we have

$$\begin{aligned}
[\varOmega, a] &= -[a, \varOmega] = \frac{1}{2} na\sqrt{1-e^2}\cos I \\
[\omega, a] &= -[a, \omega] = \frac{1}{2} na\sqrt{1-e^2} \\
[\sigma, a] &= -[a, \sigma] = \frac{1}{2} na \\
[\varOmega, e] &= -[e, \varOmega] = -\frac{na^2 e}{\sqrt{1-e^2}}\cos I \\
[\omega, e] &= -[e, \omega] = -\frac{na^2 e}{\sqrt{1-e^2}} \\
[\varOmega, I] &= -[I, \varOmega] = -na^2\sqrt{1-e^2}\sin I.
\end{aligned}$$

All other brackets are equal to zero. In obtaining the derivatives, Kepler's law of areas may be used for calculating the time derivative of the true anomaly:

$$\dot{f} = \frac{na^2\sqrt{1-e^2}}{r^2}.$$

Also, it must be kept in mind that the positions and velocities depend on the semi-major axis not only explicitly through the radius vector, but also through the mean motion, which is implicitly contained in the anomalies.

Equations (2.13) can be solved. The resulting set is that of Lagrange's equations for the variation of the osculating elements:.

$$\begin{aligned}
\dot{a} &= \frac{2}{na}\frac{\partial R}{\partial \sigma} \\
\dot{e} &= -\frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \omega} + \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial \sigma} \\
\dot{I} &= -\frac{1}{na^2 \sin I\sqrt{1-e^2}}\left(\frac{\partial R}{\partial \Omega} - \cos I\frac{\partial R}{\partial \omega}\right) \\
\dot{\sigma} &= -\frac{2}{na}\frac{\partial R}{\partial a} - \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e} \\
\dot{\omega} &= \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} - \frac{1}{na^2 \tan I\sqrt{1-e^2}}\frac{\partial R}{\partial I} \\
\dot{\Omega} &= \frac{1}{na^2 \sin I\sqrt{1-e^2}}\frac{\partial R}{\partial I}.
\end{aligned} \tag{2.16}$$

These equations are the first-order equations of the motion in the phases representation space $a, e, I, \sigma, \omega, \Omega$. The variations of the elements are very slow and, in the first approximation, the motion may be obtained by keeping them as constants in the right-hand side of the equations. Nevertheless, before integrating, some modifications must be made in order to avoid Poisson terms, whose coefficients are monotonic functions of time. Indeed, in the Fourier expansion of R the angle σ appears always through the mean anomaly $nt+\sigma$, among the arguments. Hence R will depend on the semi-major axis a through the coefficients and also through the arguments since n is a function of a. The series $\partial R/\partial a$ will have Poisson terms and they will give rise to unbounded perturbations in σ.

2.11 Tisserand's Transformation

In order to avoid this difficulty, a new parameter σ^{I} defined as

$$\frac{d\sigma^{\mathrm{I}}}{dt} = \frac{d\sigma}{dt} + t\frac{dn}{dt}$$

is introduced to σ. The new equations for a and σ^{I} are

$$\begin{aligned} \dot{a} &= \frac{2}{na}\frac{\partial R}{\partial \sigma^{\mathrm{I}}} \\ \dot{\sigma}^{\mathrm{I}} &= -\frac{2}{na}\left(\frac{\partial R}{\partial a}\right)_n - \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e} \end{aligned}$$

where $\left(\partial/\partial a\right)_n$ means that the derivative with respect to a is made without including the dependence through n in the arguments: only the coefficients are differentiated. Also

$$\begin{aligned} \dot{a} &= \frac{2}{na}\frac{\partial R}{\partial \ell} \\ \dot{\ell} &= n - \frac{2}{na}\frac{\partial R}{\partial a} - \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e} \end{aligned} \tag{2.17}$$

with $R = R(a, e, I, \ell, \omega, \Omega)$. These new equations demonstrate the need of a supplementary second-order equation when one uses the method of successive approximations for the integration. Indeed, to obtain an accurate solution up to the first power of the disturbing masses, it is not sufficient to introduce for n in equation (2.17) the undisturbed approximation; it will not give account of the part of nt due to the variation of the mean motion n. While using the method of successive approximations, in order to keep homogeneity in the quantities involved in equation (2.17), the improved quantity

$$n = \frac{d\rho}{dt}$$

must be substituted for the mean motion. It then follows

$$\frac{d^2\rho}{dt^2} = \frac{dn}{dt} = -\frac{3n}{2a}\frac{da}{dt}$$

or

$$\frac{d^2\rho}{dt^2} = -\frac{3}{a^2}\frac{\partial R}{\partial \ell}. \tag{2.18}$$

2.12 Small Eccentricities and Inclinations

In the motion of the Galilean satellites, two important features must be considered:

> The orbits of the Galilean satellites are very close to circles.

and

> The orbits of the Galilean satellites and the equator of Jupiter lie very closely in the same plane

In fact, the eccentricities are smaller than 0.01 and the inclination of the individual orbital planes over the planet's equator is not greater than 25 arc minutes ($\sin I < 0.01$).

These features eliminate the possibility of use of the anomalies as they depend, in their definitions, of the position of the perijoves and, for nearly circular orbits, these positions are poorly determined. We introduce the longitudes

$$\begin{aligned} \lambda &= \ell + \varpi \\ \varepsilon^{\mathrm{I}} &= \sigma^{\mathrm{I}} + \varpi \end{aligned}$$

where

$$\varpi = \omega + \Omega$$

is the longitude of the perijove of the orbit considered. The two features described above also allow us to use simplified equations in which all quantities involving squares of the eccentricities or inclinations are neglected. These simplified equations are

$$\begin{aligned} \frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial \lambda} & \frac{d\varepsilon^{\mathrm{I}}}{dt} &= -\frac{2}{na}\frac{\partial R}{\partial a} \\ \frac{de}{dt} &= -\frac{1}{na^2 e}\frac{\partial R}{\partial \varpi} & \frac{d\varpi}{dt} &= \frac{1}{na^2 e}\frac{\partial R}{\partial e} \\ \frac{dI}{dt} &= -\frac{1}{na^2 I}\frac{\partial R}{\partial \Omega} & \frac{d\Omega}{dt} &= \frac{1}{na^2 I}\frac{\partial R}{\partial I} \end{aligned} \tag{2.19}$$

and

$$\lambda = \rho + \varepsilon^{\mathrm{I}}$$

where ρ is determined from

$$\frac{d^2\rho}{dt^2} = -\frac{3}{a^2}\frac{\partial R}{\partial \lambda}. \tag{2.20}$$

In equations (2.19) and (2.20), R is a function of $a, e, I, \lambda, \varpi, \Omega$. The appearance of e and I in the denominator of some equations is inconvenient since in the dealt problem the orbits have very small eccentricities and inclinations. It is then desirable to use different equations. Introducing non-singular variables defined by

$$\begin{aligned} h &= e\sin\varpi & k &= e\cos\varpi \\ p &= I\sin\Omega & q &= I\cos\Omega \end{aligned}$$

in the last two pairs of equations in (2.19), we have

$$\begin{aligned} \frac{dh}{dt} &= \frac{1}{na^2}\frac{\partial R}{\partial k} & \frac{dk}{dt} &= -\frac{1}{na^2}\frac{\partial R}{\partial h} \\ \frac{dp}{dt} &= \frac{1}{na^2}\frac{\partial R}{\partial q} & \frac{dq}{dt} &= -\frac{1}{na^2}\frac{\partial R}{\partial p} \end{aligned} \qquad (2.21)$$

Practically, we will prefer to combine these variables in order to have the complex variables $k + \mathrm{i}h$ and $q + \mathrm{i}p$ (where $\mathrm{i} = \sqrt{-1}$). The corresponding equations are given in Sections 4.2 (eqn. 4.4) and 10.1 (eqn. 10.3), respectively.

References and Notes

- 2.1
  Editor's Note – In this chapter, *potential* is synonymous with *force-function*. It is the opposite of the potential as defined in Physics.

- 2.3
  Lagrange's reduction of the 3-body problem is explained in
  Y.Hagihara: 1970, *Celestial Mechanics*, M.I.T. Press, Cambridge, Vol.I, Chap. V.

- 2.10
  Equations (2.16) are discussed in
  F.Tisserand: 1896, *Traité de Mécanique Céleste*, Gauthier, Paris, Vol. I, Chap. X.

  D.Brouwer and G.M.Clemence: 1951, *Methods of Celestial Mechanics*, Academic Press, New York, Chap.XI.

  Editor's Note – In the Russian edition of this book,
  S.Ferraz-Mello: 1983, *Dinamika Galileievikh Sputinikov Yupitera*, Izd. Mir, Moscow,
  an equivalent set of Lagrange's variational equations is also included:

$$\begin{aligned} \dot{a} &= \frac{2}{na}\frac{\partial R}{\partial \lambda} \\ \dot{e} &= -\frac{\beta}{na^2 e}\left((1-\beta)\frac{\partial R}{\partial \lambda} + \frac{\partial R}{\partial \varpi}\right) \\ \dot{I} &= -\frac{\tan\frac{I}{2}}{na^2\beta}\left(\frac{\partial R}{\partial \lambda} + \frac{\partial R}{\partial \varpi}\right) - \frac{1}{na^2\beta\sin I}\frac{\partial R}{\partial \mathit{\Omega}} \\ \dot{\varepsilon}^{\mathrm{I}} &= -\frac{2}{na}\frac{\partial R}{\partial a} + \frac{\beta(1-\beta)}{na^2 e}\frac{\partial R}{\partial e} + \frac{\tan\frac{I}{2}}{na^2\beta}\frac{\partial R}{\partial I} \end{aligned} \qquad (2.22)$$

$$\begin{aligned}\dot{\varpi} &= \frac{\beta}{na^2 e}\frac{\partial R}{\partial e} + \frac{\tan\frac{I}{2}}{na^2\beta}\frac{\partial R}{\partial I} \\ \dot{\Omega} &= \frac{1}{na^2\beta\sin I}\frac{\partial R}{\partial I}\end{aligned}$$

where $R = R(a, e, I, \lambda, \varpi, \Omega)$ and $\beta = \sqrt{1 - e^2}$.

The corresponding Tisserand equation is

$$\frac{d^2\rho}{dt^2} = -\frac{3}{a^2}\frac{\partial R}{\partial \lambda} \tag{2.23}$$

where $\rho = \lambda - \varepsilon^{\mathrm{I}}$.

3

The Disturbing Functions

3.1 Forces acting on the Satellites

When satellites move around a central point-mass, the disturbing function of the satellite m_i has been shown to be

$$\sum_{j \neq i} G m_j \left(\frac{1}{r_{ij}} - \frac{\boldsymbol{r}_i \cdot \boldsymbol{r}_j}{r_j^3} \right). \tag{3.1}$$

For the four Galilean satellites of Jupiter, other disturbing actions must be considered: the action of the Sun and the actions due to the great oblateness (1/15) of Jupiter. The disturbing function that corresponds to the solar action, like (3.1), may be written as

$$R_{i0} = G m_0 \left(\frac{1}{r_{i0}} - \frac{\boldsymbol{r}_i \cdot \boldsymbol{r}_0}{r_0^3} \right).$$

where m_0 is the mass of the Sun, r_{i0} is the distance of the satellite from the Sun and $\boldsymbol{r}_0$ is the jovicentric position of the Sun.

The disturbing function corresponding to the oblateness of the planet may be written as

$$R_{iJ} = -GMJ_2 \frac{b^2}{r_i^3} P_2(\sin \phi_i) - GMJ_4 \frac{b^4}{r_i^5} P_4(\sin \phi_i) + \cdots \tag{3.2}$$

where M represents the mass of Jupiter, b its equatorial radius, J_2 and J_4 two numerical coefficients related to the shape of the equipotentials of Jupiter's gravitational field and ϕ_i the latitude of the satellite over Jupiter's equator. P_2 and P_4 are Legendre polynomials

$$\begin{aligned} P_2(x) &= \frac{1}{2}(3x^2 - 1) \\ P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3). \end{aligned}$$

In this book, we consider mainly the effects due to the second harmonic. The fourth harmonic may be considered exactly in the same way and it only modifies the results quantitatively. On account of $|J_4| \ll |J_2|$, these modifications are very small.

Other disturbing effects that may be taken into account come from other planets, Saturn in particular. The planets disturb the orbital motion of Jupiter and they make its orbital plane to oscillate. The oscillations of Jupiter's orbital plane result in inertial forces that affect the motion of the satellite since this plane is taken as reference plane. In addition, Saturn produces other strong effects in Jupiter's orbit: the Jupiter-Sun distance does not follow closely Kepler laws and thus Saturn acts on the Galilean satellite orbits through a modulation of the solar effects. This indirect action is increased by the near-resonant motion of planets Jupiter and Saturn.

The disturbing action of other satellites of Jupiter may be neglected: the greatest is Jupiter V (Amalthea), which moves inside the orbit of Jupiter I (Io), and which is some ten thousand times smaller than the Galilean satellites.

Effects due to an eventual oblateness of the gravitational fields of the Galilean satellites are very small compared to other effects and hence may be neglected.

The only relativistic effect which would be worth of consideration is the advance of the perijoves of the innermost satellites. However, the inner satellites move in near-circular orbits and their perijoves are poorly defined. The relativistic modification of Kepler's third law may not be detected because of the low precision involved in direct measurements of the mean distances of the satellite from the planet.

3.2 Expansion of the Solar Force-Function

To introduce the disturbing functions in Lagrange's equations they must be written as functions of the orbital elements of the satellites. This task, usually called expansion of the disturbing function, is performed in several ways: For solar action the force-function which gives the disturbing action may be written as

$$R_{i0} = \frac{Gm_0}{r_0}\left(\left(1+\frac{r_i^2}{r_0^2}-2\frac{r_i}{r_0}\cos S\right)^{-\frac{1}{2}}-\frac{r_i}{r_0}\cos S\right)$$

where S is the angle between the jovicentric directions of the Sun and the satellite. Using jovicentric coordinates,

$$\cos S = \frac{x_i x_0 + y_i y_0 + z_i z_0}{r_i r_0}. \tag{3.3}$$

R_{i0} may be expanded in the form of a Taylor series in the powers of r_i/r_0, which is fast convergent since r_i/r_0 is very small: it has a value less than $1/400$ for the fourth satellite for which r_i has the maximum value. So we have

$$R_{i0} = \sum_{p=2}^{\infty} R_{i0}^p; \qquad R_{i0}^p = \frac{Gm_0}{r_0} \left(\frac{r_i}{r_0}\right)^p P_p(\cos S)$$

where P_p are Legendre polynomials. The term $\cos S$ given by equation (3.3) is easily calculated by substituting the values of the coordinates in the elliptic motion (equations 2.10).

It should be emphasized that the introduction of the co-ordinates as defined in the elliptic motion does not mean that we are using an approximation. Indeed, in Section 2.8, the transformation of R into a function of the orbital elements is made by using the solutions of the undisturbed problem, that is, equations (2.10). Then

$$\begin{aligned}
\cos S \;=\; & \big(1 - \sin^2\frac{I_0}{2} - \sin^2\frac{I_i}{2} + \sin^2\frac{I_0}{2}\sin^2\frac{I_i}{2}\big)\cos(\theta_i - \theta_0) \\
& + \frac{1}{2}\sin I_0 \sin I_i \big(\cos(\theta_i - \theta_0 - \varOmega_i + \varOmega_0) - \cos(\theta_i + \theta_0 - \varOmega_i - \varOmega_0)\big) \\
& + \sin^2\frac{I_0}{2}\cos^2\frac{I_i}{2}\cos(\theta_i + \theta_0 - 2\varOmega_0) \\
& + \sin^2\frac{I_i}{2}\cos^2\frac{I_0}{2}\cos(\theta_i + \theta_0 - 2\varOmega_i) \\
& + \sin^2\frac{I_0}{2}\sin^2\frac{I_i}{2}\cos(\theta_i - \theta_0 - 2\varOmega_i + 2\varOmega_0)
\end{aligned}$$

where we have introduced the true longitudes

$$\theta_i = f_i + \varpi_i.$$

It is worthwhile to note that when the orbit of Jupiter is taken as reference plane ($I_0 = 0$), we get the simplified expression

$$\cos S = \cos^2\frac{I_i}{2}\cos(\theta_i - \theta_0) + \sin^2\frac{I_i}{2}\cos(\theta_i + \theta_0 - 2\varOmega_i).$$

For the Sun and the Galilean satellites, the eccentricities and inclinations are small and hence we may use the approximate relations

$$\begin{aligned}
\theta_i &= \lambda_i + 2e_i\sin(\lambda_i - \varpi_i) + \frac{5}{4}e_i^2\sin 2(\lambda_i - \varpi_i) \\
r_i &= a_i\left(1 + \frac{1}{2}e_i^2 - e_i\cos(\lambda_i - \varpi_i) - \frac{1}{2}e_i^2\cos 2(\lambda_i - \varpi_i)\right) \\
x_i &= r_i\big(\cos\theta_i + \frac{1}{2}I_i^2\sin\varOmega_i\sin(\lambda_i - \varOmega_i)\big) \qquad (3.4) \\
y_i &= r_i\big(\sin\theta_i - \frac{1}{2}I_i^2\cos\varOmega_i\sin(\lambda_i - \varOmega_i)\big) \\
z_i &= r_i I_i \sin(\lambda_i - \varOmega_i).
\end{aligned}$$

Limiting ourselves to R_{i0}^2 (since r_i/r_0 is very small), it follows

$$
\begin{aligned}
R_{i0} \;=\; & \frac{Gm_0}{a_0}(\frac{a_i}{a_0})^2 \left(\frac{1}{4} + \frac{3}{8}(e_i^2 + e_0^2) + \frac{3}{4}\cos(2\lambda_i - 2\lambda_0) \right. \\
& -\frac{1}{2}e_i \cos(\lambda_i - \varpi_i) + \frac{3}{4}e_0 \cos(\lambda_0 - \varpi_0) \\
& +\frac{3}{4}e_i \cos(2\lambda_0 - 3\lambda_i + \varpi_i) - \frac{9}{4}e_i \cos(2\lambda_0 - \lambda_i - \varpi_i) \\
& +\frac{15}{8}e_i^2 \cos(2\lambda_0 - 2\varpi_i) - \frac{3}{8}(I_i^2 + I_0^2) \\
& +\frac{3}{4}I_i I_0 \cos(\Omega_i - \Omega_0) + \frac{3}{8}I_i^2 \cos(2\lambda_0 - 2\Omega_i) \\
& \left. +\frac{3}{8}I_0^2 \cos(2\lambda_0 - 2\Omega_0) - \frac{3}{4}I_i I_0 \cos(2\lambda_0 - \Omega_i - \Omega_0) \right) .
\end{aligned} \tag{3.5}
$$

3.3 Expansion of Jupiter's Force-Function

To develop the force-function R_{iJ}, let the equator of Jupiter be considered. Let ϕ_i and ϕ_i' be the latitude of the satellites with respect to the planet's equator and the reference plane respectively (see Figure 3.1).

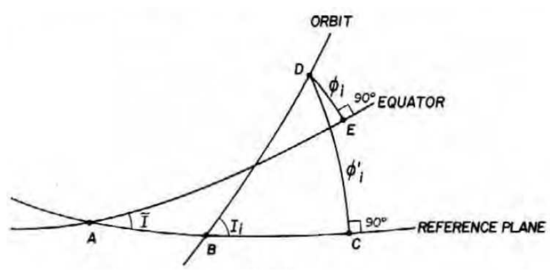

Fig. 3.1.

A and B are the ascending nodes of the equatorial and orbital planes in the reference plane, respectively, and $\widetilde{\Omega}$ and Ω_i are their longitudes reckoned from a fixed origin O. We have

$$\mathsf{BC} \simeq \mathsf{BD} \simeq \theta_i - \Omega_i \qquad\qquad \mathsf{AE} \simeq \theta_i - \widetilde{\Omega}$$

Except for quantities that are proportional to the third power of the inclinations,

$$
\begin{aligned}
\phi_i' &= I_i \sin(\theta_i - \Omega_i) \\
\phi_i' - \phi_i &= \widetilde{I} \sin(\theta_i - \widetilde{\Omega}).
\end{aligned} \tag{3.6}
$$

Thus,

$$\phi_i = I_i \sin(\theta_i - \Omega_i) - \widetilde{I} \sin(\theta_i - \widetilde{\Omega})$$

and the second harmonic of Jupiter's force-function is

$$R_{iJ} = -\frac{3}{2} GMJ_2 \frac{b^2}{r_i^3} \big(I_i \sin(\theta_i - \Omega_i) - \widetilde{I} \sin(\theta_i - \widetilde{\Omega})\big)^2 + \frac{1}{2} GMJ_2 \frac{b^2}{r_i^3}.$$

In view of equation (3.4),

$$\frac{a_i^3}{r_i^3} = 1 + 3e_i \cos(\lambda_i - \varpi_i) + \frac{3}{2} e_i^2 \big(1 + 3\cos 2(\lambda_i - \varpi_i)\big) + \cdots$$

Thus

$$\begin{aligned} R_{iJ} \ = \ & \frac{1}{2} GMJ_2 \frac{b^2}{a_i^3} \Bigg(1 + \frac{3}{2} e_i^2 + 3e_i \cos(\lambda_i - \varpi_i) + \frac{9}{2} e_i^2 \cos(2\lambda_i - 2\varpi_i) \\ & -\frac{3}{2}(I_i^2 + \widetilde{I}^2) + \frac{3}{2} I_i^2 \cos(2\lambda_i - 2\Omega_i) + \frac{3}{2} \widetilde{I}^2 \cos(2\lambda_i - 2\widetilde{\Omega}) \\ & + 3I_i \widetilde{I} \cos(\Omega_i - \widetilde{\Omega}) - 3I_i \widetilde{I} \cos(2\lambda_i - \Omega_i - \widetilde{\Omega})\Bigg). \end{aligned}$$

3.4 Laplace Coefficients

In planetary theories, the expansion of R_{ij} involves some classical procedures. The procedure described in Section 3.2 to develop R_{i0} cannot be used because the ratio of distances r_i/r_j is a large quantity and the expansion in Legendre polynomials for such case converges very slowly; this results in a large amount of terms to be considered in order to get a good approximation.

We have

$$R_{ij} = Gm_j \left(\big(r_j^2 + r_i^2 - 2r_i r_j \cos S_{ij}\big)^{-\frac{1}{2}} - \frac{r_i}{r_j^2} \cos S_{ij} \right) \tag{3.7}$$

where S_{ij} is the angle between the jovicentric directions of the two satellites involved. To expand it, we need the classical coefficients A_{ij}^k and B_{ij}^k defined by the expressions:

$$\begin{aligned} \big(a_i^2 + a_j^2 - 2a_i a_j \cos S\big)^{-\frac{1}{2}} \ &= \ \frac{1}{2} \sum_{-\infty}^{+\infty} A_{ij}^k \cos kS \\ a_i a_j \big(a_i^2 + a_j^2 - 2a_i a_j \cos S\big)^{-\frac{3}{2}} \ &= \ \frac{1}{2} \sum_{-\infty}^{+\infty} B_{ij}^k \cos kS \end{aligned} \tag{3.8}$$

and related to the Laplace coefficients. In general, for a given α $(0 < \alpha < 1)$, we define

$$\big(1 + \alpha^2 - 2\alpha \cos S\big)^{-s} = \frac{1}{2} \sum_{-\infty}^{+\infty} b_s^k \cos kS \tag{3.9}$$

where the b_s^k are the Laplace coefficients. They are the coefficients of the expansion of $(1+\alpha^2-2\alpha\cos S)^{-s}$ in a Fourier series:

$$\begin{aligned} b_s^0 &= \frac{1}{\pi}\int_0^{2\pi}\big(1+\alpha^2-2\alpha\cos S\big)^{-s}dS \\ b_s^k &= \frac{1}{\pi}\int_0^{2\pi}\big(1+\alpha^2-2\alpha\cos S\big)^{-s}\cos kS\ dS. \end{aligned} \tag{3.10}$$

To study some analytical properties of the Laplace coefficients, we introduce a new parameter z defined by

$$z = \exp \mathrm{i}S.$$

Thus,

$$\big(1+\alpha^2-2\alpha\cos S\big)^{-s} = (1-\alpha z)^{-s}(1-\alpha z^{-1})^{-s}.$$

Since $|z|=1$ and $\alpha<1$, the Taylor expansions of the terms $(1-\alpha z)^{-s}$ and $(1-\alpha z^{-1})^{-s}$ are convergent:

$$\begin{aligned} (1-\alpha z)^{-s} &= 1+\alpha s z+\frac{s(s+1)}{2!}\alpha^2 z^2 \\ &\quad +\cdots+\frac{s(s+1)\cdots(s+k-1)}{k!}\alpha^k z^k+\cdots, \\ (1-\alpha z^{-1})^{-s} &= 1+\alpha s z^{-1}+\frac{s(s+1)}{2!}\alpha^2 z^{-2} \\ &\quad +\cdots+\frac{s(s+1)\cdots(s+k-1)}{k!}\alpha^k z^{-k}+\cdots, \end{aligned}$$

and the Laplace coefficients are

$$\frac{1}{2}b_s^0 = 1+s^2\alpha^2+\left(\frac{s(s+1)}{2!}\right)^2\alpha^4+\cdots+\left(\frac{s(s+1)\cdots(s+k-1)}{k!}\right)^2\alpha^{2k}+\cdots,$$

and (3.11)

$$\begin{aligned} \tfrac{1}{2}b_s^k = \frac{s(s+1)\cdots(s+k-1)}{k!}\alpha^k\Bigg(1&+\frac{s(s+k)}{k+1}\alpha^2 \\ &+\frac{s(s+1)}{2!}\ \frac{(s+k)(s+k+1)}{(k+1)(k+2)}\alpha^4+\cdots\Bigg). \end{aligned}$$

Some useful recurrence formulae can be derived from

$$\big(1+\alpha^2-\alpha(z+\frac{1}{z})\big)^{-s} = \frac{1}{2}\sum_{-\infty}^{+\infty} b_s^k z^k. \tag{3.12}$$

Indeed, we have

$$s\alpha\big(1+\alpha^2-\alpha(z+\frac{1}{z})\big)^{-s-1}\big(1-\frac{1}{z^2}\big) = \frac{1}{2}\sum_{-\infty}^{+\infty} b_s^k k z^{k-1} \tag{3.13}$$

or

$$s\alpha \sum_{-\infty}^{+\infty} b_s^k z^k \big(1 - \frac{1}{z^2}\big) = \big(1 + \alpha^2 - \alpha(z + \frac{1}{z})\big) \sum_{-\infty}^{+\infty} b_s^k k z^{k-1}.$$

Equating the coefficients of powers of z on both sides (and shifting the subscripts inside each summation), we obtain

$$b_s^k = \frac{k-1}{k-s}\big(\alpha + \frac{1}{\alpha}\big) b_s^{k-1} - \frac{k+s-2}{k-s} b_s^{k-2}; \tag{3.14}$$

Equation (3.13) can also be written as

$$s\alpha \sum_{-\infty}^{+\infty} b_{s+1}^k z^k \big(1 - \frac{1}{z^2}\big) = \sum_{-\infty}^{+\infty} b_s^k k z^{k-1},$$

which gives

$$b_s^k = \frac{s\alpha}{k}\left(b_{s+1}^{k-1} - b_{s+1}^{k+1}\right). \tag{3.15}$$

The derivatives of b_s^k can be obtained from equation (3.12), which gives

$$-s\big(1 + \alpha^2 - \alpha(z + \frac{1}{z})\big)^{-s-1}\big(2\alpha - z - \frac{1}{z}\big) = \frac{1}{2} \sum_{-\infty}^{+\infty} \frac{db_s^k}{d\alpha} z^k;$$

equating the coefficients of powers of z on both sides we obtain

$$\frac{db_s^k}{d\alpha} = s\left(b_{s+1}^{k-1} - 2\alpha b_{s+1}^k + b_{s+1}^{k+1}\right). \tag{3.16}$$

3.5 Numerical Values for the Galilean Satellites

To get the values of the numerical coefficients that correspond to the Galilean satellites, we have to fix some constants of the motion. Adopting the values established in Section 4.6 for the osculating semi-major axes, we obtain the numerical values tabulated in Tables 3.1 to 3.9.

Table 3.1. Values of α

i-j	1-2	2-3	3-4	1-3	2-4	1-4
α	0.62844	0.62688	0.56855	0.39396	0.35642	0.22399

Table 3.2. Values of $b_{1/2}^{k}$

k	1-2	2-3	3-4	1-3	2-4	1-4
0	2.2588	2.2570	2.1998	2.0852	2.0685	2.0258
1	0.7542	0.7515	0.6558	0.4194	0.3749	0.2283
2	0.3631	0.3608	0.2843	0.1248	0.1008	0.0384
3	0.1923	0.1906	0.1358	0.0411	0.0300	0.0072
4	0.1064	0.1053	0.0679	0.0142	0.0094	0.0014
5	0.0605	0.0597	0.0349	0.0050	0.0030	0.0003
6	0.0350	0.0344	0.0182	0.0018	0.0010	0.0001

Table 3.3. Values of $b_{3/2}^{k}$

k	1-2	2-3	3-4	1-3	2-4	1-4
0	6.0172	5.9757	4.7276	2.9122	2.7085	2.2448
1	4.8810	4.8390	3.5670	1.6232	1.3804	0.7401
2	3.6170	3.5783	2.4206	0.7830	0.6048	0.2059
3	2.5699	2.5366	1.5668	0.3561	0.2494	0.0536
4	1.7814	1.7541	0.9870	0.1568	0.0995	0.0135
5	1.2148	1.1934	0.6109	0.0677	0.0389	0.0033
6	0.8189	0.8025	0.3734	0.0288	0.0150	0.0008

Table 3.4. Values of $\dfrac{db_{1/2}^{k}}{d\alpha}$

k	1-2	2-3	3-4	1-3	2-4	1-4
0	1.0996	1.0930	0.8791	0.4760	0.4151	0.2373
1	1.7497	1.7435	1.5461	1.2081	1.1646	1.0595
2	1.4524	1.4446	1.1906	0.6812	0.5993	0.3508
3	1.0842	1.0761	0.8130	0.3296	0.2633	0.0977
4	0.7729	0.7653	0.5277	0.1501	0.1087	0.0255
5	0.5367	0.5302	0.3329	0.0661	0.0434	0.0064
6	0.3664	0.3611	0.2062	0.0286	0.0169	0.0016

Table 3.5. Values of $\dfrac{d^2b_{1/2}^{k}}{d\alpha^2}$

k	1-2	2-3	3-4	1-3	2-4	1-4
0	4.2675	4.2322	3.1815	1.7041	1.5438	1.1852
1	4.0064	3.9700	2.8764	1.2588	1.0639	0.5610
2	4.9835	4.9467	3.8445	2.2703	2.0967	1.7048
3	5.2262	5.1848	3.9191	1.9038	1.6375	0.9061
4	4.8641	4.8185	3.4215	1.2402	0.9758	0.3491
5	4.1897	4.1428	2.7234	0.7122	0.5099	0.1163
6	3.4228	3.3774	2.0407	0.3792	0.2465	0.0356

Table 3.6. Values of A_{ij}^k

k	1-2	2-3	3-4	1-3	2-4	1-4
0	0.24035	0.15056	0.08343	0.13909	0.07845	0.07683
1	0.08025	0.05013	0.02487	0.02798	0.01422	0.00866
2	0.03864	0.02407	0.01078	0.00832	0.00382	0.00146
3	0.02046	0.01271	0.00515	0.00274	0.00114	0.00027
4	0.01133	0.00702	0.00258	0.00095	0.00036	0.00005
5	0.00644	0.00398	0.00132	0.00034	0.00011	0.00001
6	0.00372	0.00229	0.00069	0.00012	0.00004	

Table 3.7. Values of B_{ij}^k

k	1-2	2-3	3-4	1-3	2-4	1-4
0	0.40237	0.24988	0.10194	0.07653	0.03661	0.01907
1	0.32639	0.20235	0.07691	0.04266	0.01866	0.00629
2	0.24187	0.14963	0.05219	0.02058	0.00818	0.00175
3	0.17185	0.10607	0.03378	0.00936	0.00337	0.00046
4	0.11912	0.07335	0.02128	0.00412	0.00134	0.00012
5	0.08123	0.04990	0.01317	0.00178	0.00052	0.00003
6	0.05476	0.03356	0.00805	0.00076	0.00020	0.00001

Table 3.8. Values of $a_i \dfrac{\partial A_{ij}^k}{\partial a_i}$ $(a_i < a_j)$

k	1-2	2-3	3-4	1-3	2-4	1-4
0	0.07353	0.04570	0.01895	0.01250	0.00561	0.00201
1	0.11700	0.07291	0.03335	0.03175	0.01574	0.00900
2	0.09712	0.06041	0.02567	0.01790	0.00810	0.00298
3	0.07250	0.04499	0.01753	0.00866	0.00356	0.00083
4	0.05168	0.03200	0.01138	0.00395	0.00147	0.00022
5	0.03589	0.02217	0.00718	0.00174	0.00058	0.00005
6	0.02450	0.01510	0.00445	0.00075	0.00023	0.00001

Table 3.9. Values of $a_i^2 \dfrac{\partial^2 A_{ij}^k}{\partial a_i^2}$ $(a_i < a_j)$

k	1-2	2-3	3-4	1-3	2-4	1-4
0	0.17932	0.11094	0.03900	0.01764	0.00744	0.00226
1	0.16836	0.10407	0.03526	0.01303	0.00512	0.00107
2	0.20942	0.12967	0.04713	0.02350	0.01010	0.00324
3	0.21962	0.13591	0.04804	0.01971	0.00789	0.00172
4	0.20441	0.12631	0.04194	0.01284	0.00470	0.00066
5	0.17607	0.10860	0.03338	0.00737	0.00246	0.00022
6	0.14383	0 08853	0.02502	0.00393	0.00119	0.00007

3.6 The Force-Function of the Mutual Interactions

The disturbing function R_{ij} defined by equation (3.7) consists of two parts, which depend only on the coordinates of the two satellites. These parts are:

$$r_{ij}^{-1} = \left(r_i^2 + r_j^2 - 2r_i r_j \cos S_{ij}\right)^{-\frac{1}{2}}$$

$$Q = \frac{r_i}{r_j^2} \cos S_{ij}$$

and they can be expressed in power series of the eccentricities and inclinations; the computations are confined up to first degree terms and to second-degree longitude independent terms.

Following section 3.2 we have

$$\begin{aligned}
\cos S_{ij} \;=\; & \big(1 - \sin^2 \frac{I_j}{2} - \sin^2 \frac{I_i}{2} + \sin^2 \frac{I_j}{2} \sin^2 \frac{I_i}{2}\big) \cos(\theta_i - \theta_j) \\
& + \frac{1}{2} \sin I_i \sin I_j \big(\cos(\theta_i - \theta_j - \Omega_i + \Omega_j) - \cos(\theta_i + \theta_j - \Omega_i - \Omega_j)\big) \\
& + \sin^2 \frac{I_i}{2} \cos^2 \frac{I_j}{2} \cos(\theta_i + \theta_j - 2\Omega_i) \\
& + \sin^2 \frac{I_j}{2} \cos^2 \frac{I_i}{2} \cos(\theta_i + \theta_j - 2\Omega_j) \qquad (3.17) \\
& + \sin^2 \frac{I_j}{2} \sin^2 \frac{I_i}{2} \cos(\theta_i - \theta_j - 2\Omega_i + 2\Omega_j)
\end{aligned}$$

It is worth noting that $\cos S_{ij}$ is equal to $\cos(\theta_i - \theta_j)$ except for quantities that are at least proportional to the square of the inclinations. We can write

$$\begin{aligned}
r_{ij}^{-1} \;=\; & \big(r_i^2 + r_j^2 - 2r_i r_j \cos(\theta_i - \theta_j)\big)^{-1/2} \qquad (3.18) \\
& + r_i r_j \big(r_i^2 + r_j^2 - 2r_i r_j \cos(\theta_i - \theta_j)\big)^{-3/2} \big(\cos S_{ij} - \cos(\theta_i - \theta_j)\big).
\end{aligned}$$

In order to introduce the Laplace coefficients, we may consider the Taylor expansions:

$$\begin{aligned}
\big(r_i^2 + r_j^2 - 2r_i r_j \cos(\theta_i - \theta_j)\big)^{-1/2} &= \rho_{ij} + \frac{\partial \rho_{ij}}{\partial a_i}(r_i - a_i) + \frac{\partial \rho_{ij}}{\partial a_j}(r_j - a_j) \\
&+ \frac{1}{2} \frac{\partial^2 \rho_{ij}}{\partial a_i^2}(r_i - a_i)^2 + \frac{1}{2} \frac{\partial^2 \rho_{ij}}{\partial a_j^2}(r_j - a_j)^2 + \frac{\partial^2 \rho_{ij}}{\partial a_i \partial a_j}(r_i - a_i)(r_j - a_j)
\end{aligned}$$

where, for simplicity, we put

$$\rho_{ij} = \big(a_i^2 + a_j^2 - 2a_i a_j \cos(\theta_i - \theta_j)\big)^{-1/2}.$$

We also consider the limited expansions

$$r_i - a_i = -a_i e_i \cos(\lambda_i - \varpi_i) + \frac{1}{2} a_i e_i^2 - \frac{1}{2} a_i e_i^2 \cos 2(\lambda_i - \varpi_i)$$

and

$$\begin{aligned}\cos k(\theta_i - \theta_j) = \cos k(\lambda_i - \lambda_j) &- k e_i \cos\big((k-1)\lambda_i - k\lambda_j + \varpi_i\big) \\ &+ k e_i \cos\big((k+1)\lambda_i - k\lambda_j - \varpi_i\big) + k e_j \cos\big(k\lambda_i - (k+1)\lambda_j + \varpi_j\big) \\ &- k e_j \cos\big(k\lambda_i - (k-1)\lambda_j - \varpi_j\big) + I_{ij}^k \end{aligned} \tag{3.19}$$

where I_{ij}^k are longitude independent terms of second degree:

$$\begin{aligned} I_{ij}^k &= e_i e_j \cos(\varpi_i - \varpi_j) && \text{if } |k| = 1 \\ I_{ij}^k &= 0 && \text{if } |k| \neq 1. \end{aligned}$$

It then follows

$$\begin{aligned} \big(r_i^2 + r_j^2 - 2 r_i r_j \cos(\theta_i - \theta_j)\big)^{-1/2} \;=\; & \frac{1}{2} \sum_{-\infty}^{+\infty} A_{ij}^k \cos k(\lambda_i - \lambda_j) \\ + \frac{1}{2} e_i \sum_{-\infty}^{+\infty} & \left(2k A_{ij}^k - a_i \frac{\partial A_{ij}^k}{\partial a_i}\right) \cos\big((k+1)\lambda_i - k\lambda_j - \varpi_i\big) \\ + \frac{1}{2} e_j \sum_{-\infty}^{+\infty} & \left(2k A_{ij}^k - a_j \frac{\partial A_{ij}^k}{\partial a_j}\right) \cos\big(k\lambda_i - (k+1)\lambda_j + \varpi_j\big) \\ + e_i e_j & \left(A_{ij}^1 + \frac{a_i}{2} \frac{\partial A_{ij}^1}{\partial a_i} + \frac{a_j}{2} \frac{\partial A_{ij}^1}{\partial a_j} + \frac{a_i a_j}{4} \frac{\partial^2 A_{ij}^1}{\partial a_i \partial a_j}\right) \cos(\varpi_i - \varpi_j) \\ + \frac{1}{4} a_i e_i^2 & \left(\frac{\partial A_{ij}^0}{\partial a_i} + \frac{a_i}{2} \frac{\partial^2 A_{ij}^0}{\partial a_i^2}\right) + \frac{1}{4} a_j e_j^2 \left(\frac{\partial A_{ij}^0}{\partial a_j} + \frac{a_j}{2} \frac{\partial^2 A_{ij}^0}{\partial a_j^2}\right). \end{aligned} \tag{3.20}$$

In the calculations, the summations are applied to k from $-\infty$ to $+\infty$ and when needed k was interchanged with $-k$. This technique allows us to identify several terms and to have a more concise result. In the same way, the remaining part of the equation (3.18) reduces to

$$-\frac{1}{8} B_{ij}^1 \big(I_i^2 + I_j^2 - 2 I_i I_j \cos(\Omega_i - \Omega_j)\big). \tag{3.21}$$

The combination of equations (3.20) and (3.21) gives the expansion of r_{ij}^{-1}.

We will now develop the other part of the disturbing function. From equations (3.4), (3.17) and (3.19), it is evident that

$$\begin{aligned} -\frac{r_i}{r_j^2} \cos S_{ij} \;=\; & \frac{a_i}{a_j^2} \Big(-\cos(\lambda_i - \lambda_j) + \frac{3}{2} e_i \cos(\lambda_j - \varpi_i) \\ & - \frac{1}{2} e_i \cos(2\lambda_i - \lambda_j - \varpi_i) - 2 e_j \cos(2\lambda_j - \lambda_i - \varpi_j)\Big). \end{aligned} \tag{3.22}$$

The final expansion is then given by

$$\boxed{R_{ij} = Gm_j\big(\text{eqn.(3.20)} + \text{eqn.(3.21)} + \text{eqn.(3.22)}\big)}$$

Souillart in his study of the theory of Laplace has shown that because of the small divisors $n_1 - 2n_2$ and $n_2 - 2n_3$, it is necessary to keep in equations (3.20) and (3.21) the second-degree terms the arguments of which are $4\lambda_2 - 2\lambda_1$ and $4\lambda_3 - 2\lambda_2$. They are

$$\begin{aligned}
&\frac{1}{8}e_i^2\left(44A_{ij}^4 + 14a_i\frac{\partial A_{ij}^4}{\partial a_i} + a_i^2\frac{\partial^2 A_{ij}^4}{\partial a_i^2}\right)\cos(4\lambda_j - 2\lambda_i - 2\varpi_i)\\
&+\frac{1}{4}e_ie_j\left(-36A_{ij}^3 - 6a_i\frac{\partial A_{ij}^3}{\partial a_i} + 6a_j\frac{\partial A_{ij}^3}{\partial a_j} + a_ia_j\frac{\partial^2 A_{ij}^3}{\partial a_i\partial a_j}\right)\cos(4\lambda_j - 2\lambda_i - \varpi_i - \varpi_j)\\
&+\frac{1}{8}e_j^2\left(26A_{ij}^2 - 10a_j\frac{\partial A_{ij}^2}{\partial a_j} + a_j^2\frac{\partial^2 A_{ij}^2}{\partial a_j^2}\right)\cos(4\lambda_j - 2\lambda_i - 2\varpi_j)\\
&+\frac{1}{8}I_i^2B_{ij}^3\cos(4\lambda_j - 2\lambda_i - 2\Omega_i) + \frac{1}{8}I_j^2B_{ij}^3\cos(4\lambda_j - 2\lambda_i - 2\Omega_j)\\
&-\frac{1}{4}I_iI_jB_{ij}^3\cos(4\lambda_j - 2\lambda_i - \Omega_i - \Omega_j).
\end{aligned} \tag{3.23}$$

On the other hand, the 3:7 commensurability of Jupiter III (Ganymede) and Jupiter IV (Callisto) must also be considered. In 1892, von Haerdtl showed that the mean motions of these satellites are such that $3n_3 - 7n_4$ is a small divisor (0.044676 degrees per day), which causes a significant increase in the amplitudes of the corresponding inequalities. The main terms in the disturbing function with this argument are:

$$\begin{aligned}
&\frac{Gm_j}{384}e_4^4\left(12085A_{34}^3 + 5884a_3\frac{\partial A_{34}^3}{\partial a_3} + 894a_3^2\frac{\partial^2 A_{34}^3}{\partial a_3^2} + 52a_3^3\frac{\partial^3 A_{34}^3}{\partial a_3^3} + a_3^4\frac{\partial^4 A_{34}^3}{\partial a_3^4}\right)\\
&\qquad\cdot\cos(7\lambda_4 - 3\lambda_3 - 4\varpi_4)\\
&-\frac{Gm_j}{96}e_4^3e_3\left(11768A_{34}^4 + 5494a_3\frac{\partial A_{34}^4}{\partial a_3} + 837a_3^2\frac{\partial^2 A_{34}^4}{\partial a_3^2} + 50a_3^3\frac{\partial^3 A_{34}^4}{\partial a_3^3} + a_3^4\frac{\partial^4 A_{34}^4}{\partial a_3^4}\right)\\
&\qquad\cdot\cos(7\lambda_4 - 3\lambda_3 - \varpi_3 - 3\varpi_4)\\
&+\frac{Gm_j}{64}e_4^2e_3^2\left(11175A_{34}^5 + 5100a_3\frac{\partial A_{34}^5}{\partial a_3} + 782a_3^2\frac{\partial^2 A_{34}^5}{\partial a_3^2} + 48a_3^3\frac{\partial^3 A_{34}^5}{\partial a_3^3} + a_3^4\frac{\partial^4 A_{34}^5}{\partial a_3^4}\right)\\
&\qquad\cdot\cos(7\lambda_4 - 3\lambda_3 - 2\varpi_3 - 2\varpi_4)
\end{aligned} \tag{3.24}$$

3.7 Some Simplifications

The equations of Section 3.6 may be simplified if some classical results on Laplace coefficients are used. We may exclude all derivatives with respect to

a_j; we have

$$A_{ij}^k = \frac{1}{a_j} b_{1/2}^k \qquad B_{ij}^k = \frac{\alpha}{a_j} b_{3/2}^k \qquad (a_i < a_j)$$

then

$$\frac{\partial A_{ij}^k}{\partial a_i} = \frac{1}{a_j^2} \frac{db_{1/2}^k}{d\alpha} \tag{3.25}$$

and

$$\frac{\partial A_{ij}^k}{\partial a_j} = -\frac{1}{a_j^2}\left(\alpha \frac{db_{1/2}^k}{d\alpha} + b_{1/2}^k\right);$$

Then

$$A_{ij}^k + a_j \frac{\partial A_{ij}^k}{\partial a_j} + a_i \frac{\partial A_{ij}^k}{\partial a_i} = 0 \tag{3.26}$$

and

$$a_i a_j \frac{\partial^2 A_{ij}^k}{\partial a_i \partial a_j} = -2a_i \frac{\partial A_{ij}^k}{\partial a_i} - a_i^2 \frac{\partial^2 A_{ij}^k}{\partial a_i^2};$$

$$a_j^2 \frac{\partial^2 A_{ij}^k}{\partial a_j^2} = 2A_{ij}^k + 4a_i \frac{\partial A_{ij}^k}{\partial a_i} + a_i^2 \frac{\partial^2 A_{ij}^k}{\partial a_i^2}.$$

The equation (3.16) allow us to write the formula

$$\frac{d^2 b_{1/2}^k}{d\alpha^2} = -b_{3/2}^k + \frac{3}{4}\big(b_{5/2}^{k-2} - 4\alpha b_{5/2}^{k-1} + (2+4\alpha^2) b_{5/2}^k - 4\alpha b_{5/2}^{k+1} + b_{5/2}^{k+2}\big),$$

which has been used to get the values shown in Table 3.5. Combining equation (3.15) with

$$b_{3/2}^k = \frac{3}{2k-3}\big(2\alpha b_{5/2}^{k-1} - (1+\alpha^2) b_{5/2}^k\big)$$

we obtain, for k=0 and k=1, the following important relation

$$\alpha^2 \frac{d^2 b_{1/2}^k}{d\alpha^2} + 2\alpha \frac{db_{1/2}^k}{d\alpha} - k(k+1) b_{1/2}^k = \alpha b_{3/2}^{k+1},$$

that is,

$$a_i^2 \frac{\partial^2 A_{ij}^k}{\partial a_i^2} + 2a_i \frac{\partial A_{ij}^k}{\partial a_i} - k(k+1) A_{ij}^k = B_{ij}^{k+1}.$$

For $k = 0$ and $k = 1$, the left-hand side of this equation appears in equation (3.20) that may be written as

$$
\begin{aligned}
\left(r_i^2 + r_j^2 - 2 r_i r_j \cos(\theta_i - \theta_j)\right)^{-1/2} &= \frac{1}{2} \sum_{-\infty}^{+\infty} A_{ij}^k \cos k(\lambda_i - \lambda_j) \\
&+ \frac{1}{2} e_i \sum_{-\infty}^{+\infty} \left(2k A_{ij}^k - a_i \frac{\partial A_{ij}^k}{\partial a_i} \right) \cos\left((k+1)\lambda_i - k\lambda_j - \varpi_i\right) \\
&+ \frac{1}{2} e_j \sum_{-\infty}^{+\infty} \left((2k+1) A_{ij}^k + a_i \frac{\partial A_{ij}^k}{\partial a_i} \right) \cos\left(k\lambda_i - (k+1)\lambda_j + \varpi_j\right) \\
&+ \frac{1}{8} B_{ij}^1 (e_i^2 + e_j^2) - \frac{1}{4} B_{ij}^2 e_i e_j \cos(\varpi_i - \varpi_j). \qquad (3.27)
\end{aligned}
$$

The terms introduced by Souillart in equation (3.20) may be written as

$$
\begin{aligned}
&\frac{1}{8} e_i^2 \left(44 A_{ij}^4 + 14 a_i \frac{\partial A_{ij}^4}{\partial a_i} + a_i^2 \frac{\partial^2 A_{ij}^4}{\partial a_i^2} \right) \cos(4\lambda_j - 2\lambda_i - 2\varpi_i) \\
&-\frac{1}{4} e_i e_j \left(42 A_{ij}^3 + 14 a_i \frac{\partial A_{ij}^3}{\partial a_i} + a_i^2 \frac{\partial^2 A_{ij}^3}{\partial a_i^2} \right) \cos(4\lambda_j - 2\lambda_i - \varpi_i - \varpi_j) \\
&+\frac{1}{8} e_j^2 \left(38 A_{ij}^2 + 14 a_i \frac{\partial A_{ij}^2}{\partial a_i} + a_i^2 \frac{\partial^2 A_{ij}^2}{\partial a_i^2} \right) \cos(4\lambda_j - 2\lambda_i - 2\varpi_j). \qquad (3.28)
\end{aligned}
$$

In equations (3.27) and (3.28), it must be kept in mind that i stands for the inner satellite and j for the outer satellite· As r_{ij} is symmetrical with respect to these indices, they are interchanged in such a way that the derivatives are always made with respect to the semi-major axis of the inner satellite.

In a straightforward way, the main Laplace coefficients may be obtained from tables of elliptic integrals and from the recurrence formula (3.14). From equation (3.10) we have

$$
b_{1/2}^0 = \frac{1}{\pi} \int_0^{2\pi} (1 + \alpha^2 - 2\alpha \cos S)^{-1/2} dS
$$

$$
b_{1/2}^1 = \frac{1}{\pi} \int_0^{2\pi} (1 + \alpha^2 - 2\alpha \cos S)^{-1/2} \cos S dS
$$

A new variable defined by the transformation of Landen:

$$
\cos S = \alpha \sin^2 x + \cos x \sqrt{1 - \alpha^2 \sin^2 x}
$$

leads to

$$
\sin S = -\alpha \sin x \cos x + \sin x \sqrt{1 - \alpha^2 \sin^2 x}
$$

and

$$
(1 + \alpha^2 - 2\alpha \cos S)^{1/2} = -\alpha \cos x + \sqrt{1 - \alpha^2 \sin^2 x}
$$

and, thus, $b_{1/2}^0$ and $b_{1/2}^1$ become

$$b_{1/2}^0 = \frac{4}{\pi}\int_0^{\pi/2} \frac{dx}{\sqrt{1-\alpha^2\sin^2 x}} = \frac{4}{\pi}K(\alpha)$$

$$b_{1/2}^1 = \frac{4}{\pi}\int_0^{\pi/2} \frac{\alpha\sin^2 x dx}{\sqrt{1-\alpha^2\sin^2 x}} = \frac{4}{\pi}\frac{K(\alpha)-E(\alpha)}{\alpha}$$

where $K(\alpha)$ and $E(\alpha)$ are complete elliptic integrals of the first and second kind respectively. It is important to note that round-off errors propagates very fast in the recurrence formulae and that where precision is required the series (3.11) must be preferred.

References and Notes

- 3.1
  Effects due to the oblateness of the satellites and relativistic effects are discussed in
  S.Ferraz-Mello: 1966, "Recherches sur le Mouvement des Satellites Galiléens de Jupiter", *Bulletin Astronomique*, Série 3, **1**, 287-330.

- 3.3
  For applications, we have also considered some second-order effects of the planet's potential calculated by
  H.Kinoshita: 1977, "Third-Order Solution of an Artificial-Satellite Theory" *Smiths. Astrophys. Obs. Spec. Rep.* **379**.

- 3.4
  In many classical texts, superscripts are always put inside brackets. This is not the case in this book. The difference between powers and superscripts is always evident and an additional graphic distinction did not seem to be necessary.

- 3.6
  Some modifications follow
  O.Dziobek: 1962, *Mathematical Theory of Planetary Motions*, Dover reprint, New York, Section 26.
  (Note that Dziobek's expression for H_λ in equation (3) needs the correction of a signal).

  The terms in eqn. (3.24) are from
  G.de Pontécoulant: 1834, *Théorie Analytique du Système du Monde*, Bachelier, Paris, Vol.III, p.33.

- 3.7
  Editor's Note – The equations given at end of page 35, used to introduce B_{ij}^{k+1} in equation (3.27), were proved, for $k=0$ and $k=1$, by

F.Tisserand: 1896, *Traité de Mécanique Céleste*, Gauthier, Paris, Vol. I, p. 406.

Their validity for other values of k is not proved. They are also found in D.Brouwer and G.M.Clemence: 1951, *Methods of Celestial Mechanics*, Academic Press, New York, p. 509.

4

Inequalities of Planetary Type

4.1 Variations in Semi-major Axis and Mean Longitude

Short periodic inequalities of planetary type arise from the main part of the mutual interactions force-function:

$$
\begin{aligned}
&\frac{1}{2}Gm_j \sum A_{ij}^k \cos k(\lambda_i - \lambda_j) \qquad (4.1)\\
&+ \frac{1}{2}Gm_j e_i \sum \left(2kA_{ij}^k - a_i \frac{\partial A_{ij}^k}{\partial a_i}\right) \cos\left[(k+1)\lambda_i - k\lambda_j - \varpi_i)\right]\\
&+ \frac{1}{2}Gm_j e_j \sum \left((2k+1)A_{ij}^k + a_i \frac{\partial A_{ij}^k}{\partial a_i}\right) \cos\left[k\lambda_i - (k+1)\lambda_j + \varpi_j\right]\\
&+ Gm_j \frac{a_i}{a_j^2}\left[-\cos(\lambda_i - \lambda_j) + \frac{3}{2}e_i \cos(\lambda_j - \varpi_i)\right.\\
&\qquad \left. -\frac{1}{2}e_i \cos(2\lambda_i - \lambda_j - \varpi_i) - 2e_j \cos(2\lambda_j - \lambda_i - \varpi_j)\right],
\end{aligned}
$$

which may be written as

$$R_{ij}^0 + R_{ij}^i + R_{ij}^j$$

where R_{ij}^0, R_{ij}^i and R_{ij}^j denote the eccentricity independent terms, the terms that depend on e_i and the terms that depend on e_j, respectively. In the summations in (4.1), the constant term $\frac{1}{2}Gm_j A_{ij}^0$ and the terms whose argument comprises $2\lambda_2 - \lambda_1$ or $2\lambda_3 - \lambda_2$ are excluded. The term $k = 0$ in R_{ij}^i is also excluded.

In a first-order theory the effect of each term may be considered separately. R_{ij}^0 contributes only to the first pair of Lagrange equations and to the complementary equation (2.20). We have

$$\frac{da_i}{dt} = \frac{2}{n_i a_i}\frac{\partial R_{ij}^0}{\partial \lambda_i} \qquad \frac{d\varepsilon_i^{\mathrm{I}}}{dt} = -\frac{2}{n_i a_i}\frac{\partial R_{ij}^0}{\partial a_i}$$

and

$$\frac{d^2\rho_i}{dt^2} = -\frac{3}{a_i^2}\frac{\partial R_{ij}^0}{\partial \lambda_i}.$$

Then

$$\frac{da_i}{dt} = -\frac{Gm_j}{n_i a_i}\sum kA_{ij}^k \sin k(\lambda_i - \lambda_j) + \frac{2Gm_j}{n_i a_j^2}\sin(\lambda_i - \lambda_j)$$
$$\frac{d\varepsilon_i^{\mathrm{I}}}{dt} = -\frac{Gm_j}{n_i a_i}\sum \frac{\partial A_{ij}^k}{\partial a_i}\cos k(\lambda_i - \lambda_j) + \frac{2Gm_j}{n_i a_i a_j^2}\cos(\lambda_i - \lambda_j)$$
$$\frac{d^2\rho_i}{dt^2} = \frac{3}{2}\frac{Gm_j}{a_i^2}\sum kA_{ij}^k \sin k(\lambda_i - \lambda_j) - \frac{3Gm_j}{a_i a_j^2}\sin(\lambda_i - \lambda_j).$$

We assume, in the right-hand sides, a Keplerian approximation: all the elements, except the longitudes, are constant, and the longitudes are linear functions of the time ($\lambda = nt + \varepsilon$). The result of the integration is

$$\delta a_i = -\frac{Gm_j}{n_i a_i}\sum \frac{A_{ij}^k \cos k(\lambda_i - \lambda_j)}{n_i - n_j} - \frac{2Gm_j}{n_i a_j^2}\frac{\cos(\lambda_i - \lambda_j)}{n_i - n_j}$$
$$\delta\varepsilon_i^{\mathrm{I}} = -\frac{Gm_j}{n_i a_i}\sum \frac{\partial A_{ij}^k}{\partial a_i}\frac{\sin k(\lambda_i - \lambda_j)}{k(n_i - n_j)} + \frac{2Gm_j}{n_i a_i a_j^2}\frac{\sin(\lambda_i - \lambda_j)}{n_i - n_j}$$
$$\delta\rho_i = -\frac{3}{2}\frac{Gm_j}{a_i^2}\sum \frac{A_{ij}^k \sin k(\lambda_i - \lambda_j)}{k(n_i - n_j)^2} + \frac{3Gm_j}{a_i a_j^2}\frac{\sin(\lambda_i - \lambda_j)}{(n_i - n_j)^2}.$$

4.2 Variations in Eccentricity and Perijove

R_{ij}^i contributes to the first two pairs of Lagrange equations. The contribution to the first pair of equations is very small and may be ignored; indeed, the eccentricity of the disturbed satellite is a factor in all the results, and in the Galilean system the eccentricities are very small. The only terms which may be of interest in constructing a theory with the purpose of having good ephemerides are those which include e_4 (0.007). However, if they are considered, we have to consider also the perturbation in e_i and ϖ_i arising from terms of second degree in the eccentricities.

Since the eccentricity of the Galilean satellites is extremely small, the second pair of Lagrange equations will be considered in its modified form:

$$\frac{dh_i}{dt} = \frac{1}{n_i a_i^2}\frac{\partial R_{ij}^i}{\partial k_i} \qquad \frac{dk_i}{dt} = -\frac{1}{n_i a_i^2}\frac{\partial R_{ij}^i}{\partial h_i} \tag{4.2}$$

and

$$\begin{aligned} R^i_{ij} &= \frac{1}{2}Gm_j \sum \left(2kA^k_{ij} - a_i \frac{\partial A^k_{ij}}{\partial a_i}\right) \\ &\qquad \cdot \Big(k_i \cos[(k+1)\lambda_i - k\lambda_j] + h_i \sin[(k+1)\lambda_i - k\lambda_j]\Big) \\ &\quad + \frac{3}{2}Gm_j \frac{a_i}{a_j^2} \left(k_i \cos\lambda_j + h_i \sin\lambda_j\right) \\ &\quad - \frac{1}{2}Gm_j \frac{a_i}{a_j^2} \Big[k_i \cos(2\lambda_i - \lambda_j) + h_i \sin(2\lambda_i - \lambda_j)\Big]. \end{aligned} \tag{4.3}$$

The summation in equation (4.3) was written under the assumption that i represents the inner satellite; otherwise, inside the summation, subscripts i and j must permute.

For simplification, we introduce the complex parameter

$$\zeta_j = k_j + \mathrm{i}h_j \qquad \mathrm{i} = \sqrt{-1}.$$

Equations (4.2) become

$$\frac{d\zeta_i}{dt} = -\frac{1}{n_i a_i^2}\left(\frac{\partial R^i_{ij}}{\partial h_i} - \mathrm{i}\frac{\partial R^i_{ij}}{\partial k_i}\right)$$

or

$$\frac{d\zeta_i}{dt} = \frac{\mathrm{i}}{n_i a_i^2}\left(\frac{\partial R^i_{ij}}{\partial e_i} + \frac{\mathrm{i}}{e_i}\frac{\partial R^i_{ij}}{\partial \varpi_i}\right)\exp \mathrm{i}\varpi_i. \tag{4.4}$$

Then we have

$$\begin{aligned} \frac{d\zeta_i}{dt} &= \frac{1}{2}\frac{Gm_j}{n_i a_i^2} \sum \left(2kA^k_{ij} - a_i \frac{\partial A^k_{ij}}{\partial a_i}\right) \mathrm{i} \exp \mathrm{i}[(k+1)\lambda_i - k\lambda_j] \\ &\quad + \frac{1}{2}\frac{Gm_j}{n_i a_i a_j^2}\Big[3\mathrm{i}\exp \mathrm{i}\lambda_j - \mathrm{i}\exp \mathrm{i}(2\lambda_i - \lambda_j)\Big]. \end{aligned}$$

On integration, we get

$$\begin{aligned} \delta\zeta_i &= \frac{1}{2}\frac{Gm_j}{n_i a_i^2} \sum \left(2kA^k_{ij} - a_i \frac{\partial A^k_{ij}}{\partial a_i}\right) \frac{\exp \mathrm{i}[(k+1)\lambda_i - k\lambda_j]}{(k+1)n_i - kn_j} \\ &\quad + \frac{1}{2}\frac{Gm_j}{n_i a_i a_j^2}\Big[\frac{3\exp \mathrm{i}\lambda_j}{n_j} - \frac{\exp \mathrm{i}(2\lambda_i - \lambda_j)}{2n_i - n_j}\Big]. \end{aligned}$$

R^j_{ij} will contribute only to the first pair of Lagrange equations and is negligible; indeed the eccentricity of the disturbing satellite will be a factor in all the results and it is very small.

4.3 Inequalities in Longitude and Radius Vector

From the preceding results, we may calculate the short period inequalities of planetary type, which will affect the radius vector and the longitude of the satellites. Introducing h and k in the equations (3.4) for the longitude and radius vector and limiting to first degree terms, we obtain

$$r_i = a_i - a_i k_i \cos\lambda_i - a_i h_i \sin\lambda_i$$
$$\theta_i = \rho_i + \varepsilon_i^{\mathrm{I}} + 2k_i \sin\lambda_i - 2h_i \cos\lambda_i .$$

Their differentials, to the first degree in the eccentricities, are

$$\delta r_i = \delta a_i - a_i \delta k_i \cos\lambda_i - a_i \delta h_i \sin\lambda_i$$
$$\delta\theta_i = \delta\rho_i + \delta\varepsilon_i^{\mathrm{I}} + 2\delta k_i \sin\lambda_i - 2\delta h_i \cos\lambda_i .$$

If we introduce ζ_i, it follows

$$\delta r_i = \delta a_i - a_i \Re[\delta\zeta_i \exp(-\mathrm{i}\lambda_i)] \tag{4.5}$$
$$\delta\theta_i = \delta\rho_i + \delta\varepsilon_i^{\mathrm{I}} - 2\Im[\delta\zeta_i \exp(-\mathrm{i}\lambda_i)] . \tag{4.6}$$

The short period inequalities in the radius vector and longitudes follow from the above set of results without difficulties. The numerical results are shown in Tables 4.1 and 4.2 respectively.

Table 4.1. Coefficients of $\cos k(\lambda_i - \lambda_j)$ in δr_i (in units $10^{-7} a_i$)

	Satellite I			Satellite II			Satellite III			Satellite IV		
k	1-2	1-3	1-4	2-1	2-3	2-4	3-1	3-2	3-4	3-1	4-2	4-3
1	+209	+107	+14	+729	+633	+55	+192	+398	+291	+105	+93	+980
2	+211	−146	−9	+307	+638	−63	+6	+162	−1742		+2	+178
3	−184	−18	−1	+73	−548	−7	+1	+38	− 154			+42
4	−48	−4		+25	−144	−1		+13	−41			+13
5	−18	−1		+10	− 52			+5	−14			+5

Table 4.2. Coefficients of $\sin k(\lambda_i - \lambda_j)$ in $\delta\theta_i$ (in units 10^{-7})

	Satellite I			Satellite II			Satellite III			Satellite IV		
k	1-2	1-3	1-4	2-1	2-3	2-4	3-1	3-2	3-4	3-1	4-2	4-3
1	−753	−269	−32	+237	−2271	−135	−179	+136	−934	−105	−90	+395
2	−684	+228	+14	+389	−2064	+97	+5	+205	+3188		+1	+197
3	+285	+23	+1	+78	+849	+10	+1	+41	+224			+41
4	+65	+4		+25	+194	+2		+13	+52			+13
5	+22	+1		+10	+65			+5	+17			+5

The orbit of the disturbed satellite lies inside or outside the orbit of the disturbing satellite and this fact creates a practical problem in calculation. In

Tables 3.8 and 3.9, the derivatives of A_{ij}^k are with respect to the semi-major axis of the inner orbit. In many books the cases $i < j$ and $j < i$ are dealt separately. Here we have preferred a single formulation and we use equation (3.26) to calculate derivatives of A_{ij}^k with respect to the semi-major axis of the outer orbit.

4.4 von Haerdtl's Inequalities

The set of inequalities in the longitudes arising from (3.24), known since 1892, was calculated correctly for the first time by Lieske in 1973. The only variational equation capable of giving significant results is equation (2.20); it is a second-order equation, and, on integration, the disturbing term there has to be divided twice by the small quantity $7n_4 - 3n_3$. This may counterweigh the smallness of the fourth degree of the eccentricities. Using the values given by Lieske, we may calculate the brackets of (3.24), which become

$$\begin{array}{l} 0.5605\ Gm_j e_4^4 \cos(7\lambda_4 - 3\lambda_3 - 4\varpi_4) \\ -1.4092\ Gm_j e_4^3 e_3 \cos(7\lambda_4 - 3\lambda_3 - \varpi_3 - 3\varpi_4) \\ +1.3221\ Gm_j e_4^2 e_3^2 \cos(7\lambda_4 - 3\lambda_3 - 2\varpi_3 - 2\varpi_4). \end{array}$$

The variational equations for satellites III and IV are then

$$\begin{aligned} \frac{d^2\rho_3}{dt^2} = & -0.00326 e_4^4 \sin(7\lambda_4 - 3\lambda_3 - 4\varpi_4) \\ & -0.00821 e_4^3 e_3 \sin(7\lambda_4 - 3\lambda_3 - \varpi_3 - 3\varpi_4) \\ & -0.00770 e_4^2 e_3^2 \sin(7\lambda_4 - 3\lambda_3 - 2\varpi_3 - 2\varpi_4) \end{aligned}$$

and

$$\begin{aligned} \frac{d^2\rho_4}{dt^2} = & +0.00345\ e_4^4 \sin(7\lambda_4 - 3\lambda_3 - 4\varpi_4) \\ & -0.00867\ e_4^3 e_3 \sin(7\lambda_4 - 3\lambda_3 - \varpi_3 - 3\varpi_4) \\ & +0.00813\ e_4^2 e_3^2 \sin(7\lambda_4 - 3\lambda_3 - 2\varpi_3 - 2\varpi_4). \end{aligned}$$

Before integrating these equations, we need to know the behavior of the osculating eccentricities and perijoves, which are discussed in Section 6.4. If we adopt the results given in that section the integration of the first equation gives:

$$\begin{aligned} \delta\theta_3 = \delta\rho_3 = & \ 18.5 \times 10^{-6}\ \sin(7\lambda_4 - 3\lambda_3 - 4g^4 t - 4\beta^4) \\ & -13.7 \times 10^{-6}\ \sin(7\lambda_4 - 3\lambda_3 - 3g^4 t - 3\beta^4 - g^3 t - \beta^3) \\ & +2.3 \times 10^{-6}\ \sin(7\lambda_4 - 3\lambda_3 - 2g^4 t - 2\beta^4 - 2g^3 t - 2\beta^3) \end{aligned}$$

where $g^\mu t + \beta^\mu$ are the proper perijoves (see Section 5.3). The periods of the components of the inequalities in $\delta\rho_3$ are 26.4, 31.1 and 37.8 years, respectively. There are other components but they are negligible. $\delta\rho_4$ is similar and may be obtained from

$$\delta\theta_4 = \delta\rho_4 = -1.056\ \delta\rho_3.$$

4.5 The Constant Perturbation

In Section 4.1, we neglected the case $k = 0$:

$$\sum_j \frac{1}{2} G m_j \left(A_{ij}^0 - a_i e_i \frac{\partial A_{ij}^0}{\partial a_i} \cos(\lambda_i - \varpi_i) \right)$$

Similar terms exist in the force function which corresponds to the gravitational fields of the planet and of the Sun. They are

$$\frac{1}{2} G J_2 \frac{b^2}{a_i^3} \big(1 + 3e_i \cos(\lambda_i - \varpi_i)\big)$$

and

$$\frac{Gm_0}{4a_0} (\frac{a_i}{a_0})^2 \big(1 - 2e_i \cos(\lambda_i - \varpi_i)\big).$$

The corresponding part in the variational equations are

$$\frac{da_i}{dt} = 0 \qquad\qquad \frac{d^2\rho_i}{dt^2} = 0$$

$$\frac{d\varepsilon_i^{\mathrm{I}}}{dt} = -\sum \frac{Gm_j}{n_i a_i} \frac{\partial A_{ij}^0}{\partial a_i} + \frac{3GJ_2 b^2}{n_i a_i^5} - \frac{Gm_0}{n_i a_0^3}$$

except for terms of first degree in the eccentricities, and

$$\frac{d\zeta_i}{dt} = -\left(\sum \frac{Gm_j}{2n_i a_i} \frac{\partial A_{ij}^0}{\partial a_i} - \frac{3GJ_2 b^2}{2n_i a_i^5} + \frac{Gm_0}{2n_i a_0^3} \right) \mathrm{i} \exp \mathrm{i}\lambda_i.$$

On integration, we have

$$\begin{aligned} \delta\varepsilon_i^{\mathrm{I}} &= \sigma_i n_i \delta t \\ \delta\zeta_i &= \frac{1}{2}\sigma_i \exp \mathrm{i}\lambda_i \end{aligned} \tag{4.7}$$

where

$$\sigma_i = -\sum \frac{Gm_j}{n_i^2 a_i} \frac{\partial A_{ij}^0}{\partial a_i} + \frac{3GJ_2 b^2}{n_i^2 a_i^5} - \frac{Gm_0}{n_i^2 a_0^3}. \tag{4.8}$$

Inclusion of perturbations in J_4 leads to add

$$-\frac{15}{4}\frac{GJ_4b^4}{n_i^2a_i^7}$$

to σ_i. For the Galilean satellites, we obtain the values:

$$\begin{aligned}\sigma_1 &= 0.00125\\ \sigma_2 &= 0.000603\\ \sigma_3 &= 0.000360\\ \sigma_4 &= 0.000415.\end{aligned}$$

4.6 Osculating Mean Motion and Semi-major Axis. Mean Distance

The perturbations due to equations (4.7) in the radius vector and longitude are, respectively,

$$\delta r_i = -\frac{1}{2}a_i\sigma_i \qquad\qquad \delta\theta_i = \sigma_i n_i \delta t.$$

They are called *constant perturbations*. The perturbation in radius vector is a constant which shall be added to the osculating semi-major axis in order to obtain the mean distance from the satellite to the planet, that is

$$\overline{r}_i = a_i(1-\frac{1}{2}\sigma_i).$$

Similarly the perturbation in the longitude is proportional to time and the non-periodic variation of the longitude is

$$n_i(1+\sigma_i)\delta t.$$

Thus, the observed mean motion $\widetilde{n}_i$ is related to the osculating mean motion through the equation

$$\widetilde{n}_i = n_i(1+\sigma_i). \tag{4.9}$$

If the observed mean motion is used in the Kepler's third law, we obtain a wrong semi-major axis:

$$\widetilde{a}_i = \sqrt[3]{\frac{G(1+m_i)}{\widetilde{n}_i^2}}.$$

However the correct value is obtained by using the osculating mean motion:

$$a_i = \sqrt[3]{\frac{G(1+m_i)}{n_i^2}}.$$

To a first order approximation, it results

$$a_i = \widetilde{a}_i(1 + \frac{2}{3}\sigma_i)$$

$$\overline{r}_i = \widetilde{a}_i(1 + \frac{1}{6}\sigma_i).$$

If these rules are applied to the actual values, we obtain the results shown in Table 4.3. In that table, values listed in column $\widetilde{n}_i$ are Sampson's values for the observed sidereal mean motions of the satellites. Since Sampson's unit of time is wrong, the correction

$$E.T. = t_s(1 - 1.14 \times 10^{-8}) \tag{4.10}$$

was used. This correction is based on the analysis of the observations and on Sampson's work to determine the mean motions from old observations (see Section 11.3).

The other values are obtained by means of the first-order formulae given in this section.

Table 4.3. Mean Distances and Osculating Values

i	$\widetilde{n}_i$	a_i	n_i	$\overline{r}_i$
1	3.551 552 280	5.9060	3.547 10.	5.9023
2	1.769 322 721	9.3979	1.768 26	9.3951
3	0.878 207 942	14.992	0.877 891	14.989
4	0.376 486 223	26.368	0.376 330	26.362

It is worth noting that the observed mean motion is the coefficient of time in the equation of the mean longitude when perturbations are considered. Thus, precision is increased in computation (mainly when a small divisor exists) when the observed mean motions are used instead of osculating mean motions. However, the semi-major axes appearing in the equations should not be confused with the meaningless value $\widetilde{a}_i$ or with the mean distance from the planet $\overline{r}_i$. The osculating values a_i are not affected by secular or constant perturbations of the first order (and even of the second order), and are considered as such in computations.

References and Notes

- 4.2
  There are equations where i is used simultaneously as subscript (i) and also to represent the imaginary unit (i). The meanings are different enough and avoid confusion. On the other hand, we considered to be unnecessary to recall at every moment that i is the imaginary unit.

- 4.4
  See
  J.H.Lieske: 1973, "On the 3-7 Commensurability between Jupiter's Outer Two Galilean Satellites", *Astron. Astrophys. 27*, 59-65.

- 4.6
  Equation (4.10) corresponds to a unit correction of 0.6 minutes per century. This slope is shown by the dashed line in Figure 11.1.
  In this book, we adopted the mass and the equatorial radius of Jupiter as units of mass and length. The unit of time is the day. Thus

$$G = 2598.347\ b_0^3/\mathrm{M_J d^2}$$

5

The Equations of the Centre - I

5.1 The Variational Equations

In the preceding chapter, the terms yielding small divisors have not been included. We shall consider those terms along with the quadratic expression in eccentricities of the force-function:

$$\begin{aligned} &\frac{1}{8}Gm_jB^1_{ij}(e_i^2+e_j^2)-\frac{1}{4}Gm_jB^2_{ij}e_ie_j\cos(\varpi_i-\varpi_j) \\ &+\frac{3}{8}\frac{Gm_0}{a_0}(\frac{a_i}{a_0})^2(e_i^2+e_0^2)+\frac{3}{4}GJ_2\frac{b^2e_i^2}{a_i^3}. \end{aligned} \tag{5.1}$$

The terms of expression (4.1) which involve the critical arguments

$$\begin{aligned} u &= 2\lambda_2-\lambda_1 \\ u' &= 2\lambda_3-\lambda_2 \end{aligned} \tag{5.2}$$

are, for the three inner satellites, respectively,

$$\begin{aligned} &F_{12}n_1a_1^2e_1\cos(u-\varpi_1)+\frac{m_2}{m_1}G'_{12}n_2a_2^2e_2\cos(u-\varpi_2); \\ &F_{23}n_2a_2^2e_2\cos(u'-\varpi_2)+\frac{m_3}{m_2}G'_{23}n_3a_3^2e_3\cos(u'-\varpi_3) \\ &\qquad +G_{12}n_2a_2^2e_2\cos(u-\varpi_2)+\frac{m_1}{m_2}F_{12}n_1a_1^2e_1\cos(u-\varpi_1) \end{aligned} \tag{5.3}$$

and

$$G_{23}n_3a_3^2e_3\cos(u'-\varpi_3)+\frac{m_2}{m_3}F_{23}n_2a_2^2e_2\cos(u'-\varpi_2),$$

where

$$F_{ij} = -\frac{1}{2}\frac{Gm_j}{n_ia_i^2}\left(4A^2_{ij}+a_i\frac{\partial A^2_{ij}}{\partial a_i}\right) \qquad (j=i+1)$$

$$\begin{aligned} G_{ij} &= \frac{1}{2}\frac{Gm_i}{n_j a_j^2}\left(2A_{ij}^1 - a_j\frac{\partial A_{ij}^1}{\partial a_j}\right) - \frac{1}{2}\frac{Gm_i}{n_j a_j a_i^2} \\ G'_{ij} &= \frac{1}{2}\frac{Gm_i}{n_j a_j^2}\left(2A_{ij}^1 - a_j\frac{\partial A_{ij}^1}{\partial a_j}\right) - 2\frac{Gm_i a_i}{n_j a_j^4}. \end{aligned} \tag{5.4}$$

Note that $G_{ij} \simeq G'_{ij}$ since $4a_i^3 \simeq a_j^3$ for $j = i + 1$.

The differential equations for ζ_i arising from these parts of the disturbing functions are

$$\begin{aligned} &\mathrm{i}\frac{d\zeta_1}{dt} - \{1,1\}\zeta_1 - \{1,2\}\zeta_2 - \{1,3\}\zeta_3 - \{1,4\}\zeta_4 = -F_{12}\exp\mathrm{i}u \\ &\mathrm{i}\frac{d\zeta_2}{dt} - \{2,1\}\zeta_1 - \{2,2\}\zeta_2 - \{2,3\}\zeta_3 - \{2,4\}\zeta_4 = -G_{12}\exp\mathrm{i}u - F_{23}\exp\mathrm{i}u' \\ &\mathrm{i}\frac{d\zeta_3}{dt} - \{3,1\}\zeta_1 - \{3,2\}\zeta_2 - \{3,3\}\zeta_3 - \{3,4\}\zeta_4 = -G_{23}\exp\mathrm{i}u' \\ &\mathrm{i}\frac{d\zeta_4}{dt} - \{4,1\}\zeta_1 - \{4,2\}\zeta_2 - \{4,3\}\zeta_3 - \{4,4\}\zeta_4 = 0 \end{aligned} \tag{5.5}$$

where $\mathrm{i} = \sqrt{-1}$ and

$$\begin{aligned} \{i,i\} &= -\frac{G}{n_i a_i^2}\sum_{j\neq i}\frac{1}{4}m_j B_{ij}^1 - \frac{3Gm_0}{4n_i a_0^3} - \frac{3GJ_2 b^2}{2n_i a_i^5} \\ \{i,j\} &= \frac{Gm_j B_{ij}^2}{4n_i a_i^2} \qquad (j \neq i). \end{aligned}$$

The numerical values of the elements $\{i,j\}$ are given in Table 5.1. In the diagonal elements we considered also the second-order contributions

$$-\frac{3G}{8n_i a_i^7}(10J_4 - 21J_2^2).$$

Table 5.1. The matrix $\{i,j\}$ (in units $10^{-7}\mathrm{d}^{-1}$)

	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	-23276	325	85	5
$i=2$	473	-5790	488	19
$i=3$	32	126	-1261	96
$i=4$	2	5	102	-331

The coefficients F_{ij} and G_{ij} are

$$\begin{aligned} F_{12} &= -6756 \times 10^{-8}\mathrm{d}^{-1} & F_{23} &= -10213 \times 10^{-8}\mathrm{d}^{-1} \\ G_{12} &= +3451 & G_{23} &= +\ \ 902 \\ G'_{12} &= +3527 & G'_{23} &= +\ \ 944 \end{aligned}$$

5.2 The Free Oscillations

It is noteworthy that the figures in the main diagonal of matrix $\{i, j\}$ are not always much greater than those in the same row or column. Due to this fact, we cannot accept approximated solutions obtained from the separated equations

$$\frac{d\zeta_j}{dt} + \mathrm{i}\{j, j\}\zeta_j = 0$$

as it is done in some other problems. It is not possible to disconnect the system's free oscillations. The solution is to be obtained through integration of the complete system formed by equations (5.5). The associated homogeneous system is

$$\frac{d\zeta_j}{dt} + \sum_{k=1}^{4} \mathrm{i}\{j, k\}\zeta_k = 0 \tag{5.6}$$

whose fundamental solutions are

$$\zeta_j = C_j \exp \mathrm{i}gt$$

where g are the roots of the characteristic polynomial

$$\det(g\delta_{jk} + \{j, k\}) = 0. \tag{5.7}$$

To have fundamental solutions like this, the roots of the characteristic polynomial cannot be equal and, for bounded oscillations, they must be real. Laplace showed that the roots are always real, but in general it is not possible to be sure that they are all unequal. Tisserand and Seeliger showed the nonexistence of multiple roots for systems formed by two or three orbiting bodies. Darboux, using Kronecker's theory of quadratic forms showed that there may be equal roots in the case of more than three orbiting bodies.

In case of the Galilean satellites of Jupiter, the characteristic roots are indeed not multiple and the general solution of differential equation (5.6) is

$$\zeta_j = \sum_{\mu=1}^{4} C_j^\mu \exp \mathrm{i}g^\mu t$$

where integration constants C_j^μ are complex and may be written as

$$C_j^\mu = M_j^\mu \exp \mathrm{i}\beta^\mu.$$

It then follows:

$$\zeta_j = \sum_{\mu=1}^{4} M_j^\mu \exp \mathrm{i}(g^\mu t + \beta^\mu). \tag{5.8}$$

The real constants M_j^μ are not independent and must satisfy

$$\sum_{k=1}^{4} (g^\mu \delta_{jk} + \{j, k\}) M_k^\mu = 0$$

for each value of μ.

5.3 Proper Eccentricities and Perijoves

Either in the case where we have a single body orbiting around a primary or in the case where only main diagonal elements of matrix $\{i, j\}$ need to be considered, the solution can be written as

$$\zeta = M \exp \mathfrak{i}(gt + \beta)$$

which leads to

$$\begin{aligned} \delta r &= -aM \cos(\lambda - gt - \beta) \\ \delta\theta &= 2M \sin(\lambda - gt - \beta) \end{aligned} \tag{5.9}$$

where M and β are real integration constants. These equations are the equations of the centre of Keplerian motion. By analogy, M_j^j are called *proper eccentricities*. Values of these integration constants adopted by Sampson and values determined by de Sitter and by Lieske are given in Table 5.2.

Table 5.2. Proper Eccentricities (in units 10^{-6})

	Sampson (1910)	de Sitter (1931)	Lieske (1978)
M_1^1	46	11 ± 9	10±4
M_2^2	82	131±19	92 ± 24
M_3^3	1517	1390±34	1469 ± 28
M_4^4	7373	7362±13	7333 ± 27

In equations (5.9), the angles $gt + \beta$ are the longitudes of the *proper perijoves* and their motions are the roots of the characteristic equation. Assignment of a characteristic root to a satellite is done without ambiguity. In non-coupled case, we have $g^j = -\{j, j\}$; whereas in coupled case, the result is not exactly the same but the order of magnitude is maintained. The values reported in Table 5.4 are not much different from the absolute values of the main diagonal in Table 5.1.

There are discrepancies in the values obtained here as roots of equation (5.7) and by Sampson as well as by de Sitter (see Table 5.4). This is due to the fact that the Laplace-Lagrange theory of secular perturbations is not sufficient to describe the phenomena when the mean motions are close to resonance. Thus, we will consider the great long period inequalities in the mean longitudes and equation (5.5) will accordingly be modified to get a better solution for secular perturbations.

5.4 Great Inequalities in the Mean Longitudes

The terms (5.3) of the disturbing function depend on the semi-major axes and longitudes and contribute to other variational equations, viz. the equations for ρ_i, ε_i and a_i. These contributions are of first degree in eccentricities and they do not affect much excepting when they contain the square of

$$\mathfrak{m} = n_1 - 2n_2 = n_2 - 2n_3 \tag{5.10}$$

in denominator. Thus the only equations which should be considered are the second-order equations (2.20) that give the mean longitudes:

$$\begin{aligned}
\frac{d^2\rho_1}{dt^2} &= -3n_1 F_{12} e_1 \sin(u-\varpi_1) - 3n_2 \frac{m_2}{m_1}\frac{a_2^2}{a_1^2} G'_{12} e_2 \sin(u-\varpi_2) \qquad (5.11)\\
\frac{d^2\rho_2}{dt^2} &= -3n_2 F_{23} e_2 \sin(u'-\varpi_2) - 3n_3 \frac{m_3}{m_2}\frac{a_3^2}{a_2^2} G'_{23} e_3 \sin(u'-\varpi_3)\\
&\quad +6n_2 G_{12} e_2 \sin(u-\varpi_2) + 6n_1 \frac{m_1}{m_2}\frac{a_1^2}{a_2^2} F_{12} e_1 \sin(u-\varpi_1)\\
\frac{d^2\rho_3}{dt^2} &= 6n_3 G_{23} e_3 \sin(u'-\varpi_3) + 6n_2 \frac{m_2}{m_3}\frac{a_2^2}{a_3^2} F_{23} e_2 \sin(u'-\varpi_2).
\end{aligned}$$

To include the results of Section 5.2, we introduce the formula

$$e_i \sin(\Phi - \varpi_i) = \sum M_i^\mu \sin(\Phi - g^\mu t - \beta^\mu). \tag{5.12}$$

Thus, we have

$$\begin{aligned}
\frac{d^2\rho_1}{dt^2} &= \sum_\mu H'^\mu_{12} \sin(u - g^\mu t - \beta^\mu)\\
\frac{d^2\rho_2}{dt^2} &= \sum_\mu (H^\mu_{12} - H'^\mu_{23}) \sin(u - g^\mu t - \beta^\mu) \qquad (5.13)\\
\frac{d^2\rho_3}{dt^2} &= \sum_\mu -H^\mu_{23} \sin(u - g^\mu t - \beta^\mu)
\end{aligned}$$

where

$$\begin{aligned}
H^\mu_{ij} &= 6n_j G_{ij} M_j^\mu + 6n_i \frac{m_i}{m_j}\frac{a_i^2}{a_j^2} F_{ij} M_i^\mu\\
H'^\mu_{ij} &= -3n_i F_{ij} M_i^\mu - 3n_j \frac{m_j}{m_i}\frac{a_j^2}{a_i^2} G'_{ij} M_j^\mu
\end{aligned}$$

and, to avoid lengthy formulae, the result of the theorem of Laplace, $u = u' + \pi$, described in Section 7.5, has been used.

The direct integration leads to the great inequalities in the mean longitudes:

$$\begin{aligned}
\delta\theta_1 = \delta\rho_1 &= -\sum_{\mu} \frac{H'^{\mu}_{12}}{(\mathfrak{m} + g^{\mu})^2} \sin(u - g^{\mu}t - \beta^{\mu}) \\
\delta\theta_2 = \delta\rho_2 &= -\sum_{\mu} \frac{H^{\mu}_{12} - H'^{\mu}_{23}}{(\mathfrak{m} + g^{\mu})^2} \sin(u - g^{\mu}t - \beta^{\mu}) \qquad (5.14) \\
\delta\theta_3 = \delta\rho_3 &= \sum_{\mu} \frac{H^{\mu}_{23}}{(\mathfrak{m} + g^{\mu})^2} \sin(u - g^{\mu}t - \beta^{\mu}).
\end{aligned}$$

These inequalities have large periods, ranging from 400 to 486 days, and their amplitudes are discussed in Section 7.6. Some of them are the most important in satellite's motion excepted the main equations of the centre and the annual equation of Jupiter III(Ganymede).

5.5 New Equations of the Free Oscillations

The inequalities calculated in the preceding section affect angles u and u':

$$\begin{aligned}
\delta u &= 2\delta\rho_2 - \delta\rho_1 = \sum \mathcal{A}^{\mu} \sin(u - g^{\mu}t - \beta^{\mu}) \\
\delta u' &= 2\delta\rho_3 - \delta\rho_2 = \sum \mathcal{A}'^{\mu} \sin(u - g^{\mu}t - \beta^{\mu})
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A}^{\mu} &= \frac{2H'^{\mu}_{23} - 2H^{\mu}_{12} + H'^{\mu}_{12}}{(\mathfrak{m} + g^{\mu})^2} \\
\mathcal{A}'^{\mu} &= \frac{2H^{\mu}_{23} + H^{\mu}_{12} - H'^{\mu}_{23}}{(\mathfrak{m} + g^{\mu})^2}.
\end{aligned}$$

We will write equations (5.5) in a different form. Using Taylor's theorem, we have

$$\exp \mathrm{i}u = \exp \mathrm{i}u_0 + \mathrm{i}\delta u \exp \mathrm{i}u_0$$

$$\exp \mathrm{i}u' = \exp \mathrm{i}u'_0 + \mathrm{i}\delta u' \exp \mathrm{i}u'_0$$

where u_0 and u'_0 are the undisturbed values

$$u_0 = -\mathfrak{m}t + 2\varepsilon_2 - \varepsilon_1$$

$$u'_0 = -\mathfrak{m}t + 2\varepsilon_3 - \varepsilon_2.$$

By Laplace theorem (Section 7.5) we have $\exp \mathrm{i}u'_0 = -\exp \mathrm{i}u_0$. Thus, to a first-order approximation:

$$\exp \mathfrak{i}u = \exp \mathfrak{i}u_0 - \frac{1}{2}\sum \mathcal{A}^\mu \big[\exp \mathfrak{i}(g^\mu t + \beta^\mu) - \exp \mathfrak{i}(2u_0 - g^\mu t - \beta^\mu)\big]$$

$$\exp \mathfrak{i}u' = -\exp \mathfrak{i}u_0 + \frac{1}{2}\sum \mathcal{A}'^\mu \big[\exp \mathfrak{i}(g^\mu t + \beta^\mu) - \exp \mathfrak{i}(2u_0 - g^\mu t - \beta^\mu)\big].$$

The exponential terms involving u_0 and $2u_0$ give forced oscillations, whereas the terms not containing u_0 are homogeneous with respect to the integration constants and must be considered together with the homogeneous part of the equations (5.5). Thus, instead of equations (5.6) we have

$$\begin{aligned}
\mathfrak{i}\frac{d\zeta_1}{dt} - \sum_k \{1,k\}\zeta_k &= \frac{1}{2}F_{12}\sum_\mu \mathcal{A}^\mu \exp \mathfrak{i}(g^\mu t + \beta^\mu) \\
\mathfrak{i}\frac{d\zeta_2}{dt} - \sum_k \{2,k\}\zeta_k &= \frac{1}{2}\sum_\mu \big(G_{12}\mathcal{A}^\mu - F_{23}\mathcal{A}'^\mu\big) \exp \mathfrak{i}(g^\mu t + \beta^\mu) \\
\mathfrak{i}\frac{d\zeta_3}{dt} - \sum_k \{3,k\}\zeta_k &= -\frac{1}{2}G_{23}\sum_\mu \mathcal{A}'^\mu \exp \mathfrak{i}(g^\mu t + \beta^\mu) \qquad (5.15) \\
\mathfrak{i}\frac{d\zeta_4}{dt} - \sum_k \{4,k\}\zeta_k &= 0
\end{aligned}$$

and its fundamental solutions are

$$\zeta_j = \sum M_j^\mu \exp \mathfrak{i}(g^\mu t + \beta^\mu).$$

On substitution of the fundamental solutions and identification of coefficients of equal arguments, we obtain

$$\begin{aligned}
-g^\mu M_1^\mu - \sum \{1,k\} M_k^\mu &= \frac{1}{2}F_{12}\mathcal{A}^\mu \\
-g^\mu M_2^\mu - \sum \{2,k\} M_k^\mu &= \frac{1}{2}\big(G_{12}\mathcal{A}^\mu - F_{23}\mathcal{A}'^\mu\big) \\
-g^\mu M_3^\mu - \sum \{3,k\} M_k^\mu &= -\frac{1}{2}G_{23}\mathcal{A}'^\mu \\
-g^\mu M_4^\mu - \sum \{4,k\} M_k^\mu &= 0
\end{aligned}$$

The right hand sides of these equations are linear combinations of the M_k^μ. Thus we have the homogeneous system of linear equations:

$$\sum_{k=1}^{4} \left(g^\mu \delta_{jk} + \{j,k\} + \frac{a_{jk}}{(\mathfrak{m} + g^\mu)^2} \right) M_k^\mu = 0 \qquad (5.16)$$

where the values of the a_{jk} are given in Table 5.3.

Table 5.3. The matrix (a_{jk}) in units $10^{-11}\mathrm{d}^{-3}$

	$k=1$	$k=2$	$k=3$	$k=4$
$j=1$	-9485	-315	1309	0
$j=2$	-486	-2157	806	0
$j=3$	471	205	-130	0
$j=4$	0	0	0	0

For non-trivial solutions of equation (5.16), the determinant of the matrix of coefficients must vanish. The new values of the characteristic roots are shown in Table 5.4 where in addition are shown the roots of the characteristic polynomial (5.7).

Table 5.4. Characteristic Roots (in units $10^{-6}\mathrm{d}^{-1}$

μ	Roots of Eqn. (5.16)	Roots of Eqn. (5.7)
1	2717	2329
2	699	580
3	130	126
4	32.0	32.0

It is interesting to see how the characteristic roots are formed. The contribution of every source is shown for the innermost and outermost satellites in Table 5.5.

Table 5.5. Partial contributions to the characteristic roots

Source	Jupiter I (Io)	Jupiter IV (Callisto)
Other satellites	$63.2 \times 10^{-6}\mathrm{d}^{-1}$	$16.89 \times 10^{-6}\mathrm{d}^{-1}$
Sun	0.4	4.19
Oblateness–First Order	2252.7	11.97
Oblateness–Second Order	11.3	0.003
Forced Oscillations (a_{jk})	388.5	0.04
Coupling	0.9	–1.06

Relativistic effects are much smaller than errors involved in the determination of characteristic roots and than errors involved in the observational determination ot the motion of the perijoves. For example for Jupiter I(Io) the relativistic contribution to the characteristic root is $\sim 0.036 \times 10^{-6}\mathrm{d}^{-1}$. However such effects are greater than in other motions in the Solar System.

The characteristic roots are also affected by the Libration. The effects of the Libration are considered in Section 7.7 where corrected values for the characteristic roots are obtained (Table 7.2).

5.6 The Free Equations of the Centre

The effects of the free oscillations in the longitudes and radius vector of the satellite orbits are obtained by using equations (4.5) and (4.6). It follows that

$$\delta r_i = -a_i \sum_\mu M_i^\mu \cos(\lambda_i - g^\mu t - \beta^\mu)$$

$$\delta\theta_i = 2 \sum_\mu M_i^\mu \sin(\lambda_i - g^\mu t - \beta^\mu). \tag{5.17}$$

These inequalities are characteristic of the system of Galilean satellites of Jupiter. It is noteworthy that the strong interactions among the satellites did not allow us to accept approximate solutions arising from separated equations and thus resulted four inequalities similar to the equation of the centre for each satellite. In each inequality, the oscillation is referred to the proper perijove of one of the satellites.

According the description by Laplace in book IV of its Exposition du Système du Monde *"the eccentricity of the orbit of the third satellite presents unique irregularities and the theory allowed me to know the origin. They depend on two separate equations of the centre. The first, proper to this orbit is referred to a perijove whose annual sidereal motion is 9400 arcseconds, and the other, which may be considered an emanation of the equation of the centre of the fourth satellite, is referred to the perijove of this last body. [· · ·] These two equations, when combined, lead to a variable equation of the centre referred to a perijove whose motion is not uniform. They were coincident and were added in 1682, and their sum rose to 796 arcseconds. In 1777 they were subtracted one of the other and their difference was only 307 arcseconds."*

The study of these inequalities puts some difficulties that are still not solved. The free equations of the centre of Jupiter III (Ganymede) may be written as

$$\delta\theta_3 = 2M_3^3 \sin(\lambda_3 - g^3 t - \beta^3) + 2M_3^4 \sin(\lambda_3 - g^4 t - \beta^4)$$

since M_3^1 and M_3^2 are very small. The longitude $g^4 t + \beta_4$ is well known and the observations of Jupiter III may serve to determine the remaining parameters. However, de Sitter concluded that it is impossible to find a common solution valid for the old observations of eclipses of Jupiter III (1668 to 1898) and modern extra-eclipse observations done in the first quarter of this century.

The first solution in Table 5.6 leaves large residuals in the old eclipses and the second would leave large residuals in the representation of the modern observations for which no explanation is available. If the first solution is adopted, the only possibility for the satisfactory representation of the old observations is the assumption of a systematic error depending on the zenith distance of Jupiter, or in the season of the year, having roughly the period of Jupiter's time of revolution.

Table 5.6. de Sitter results

Parameter	Modern Observations only	Old and Modern Observations
M_3^3	$139 \pm 3 \times 10^{-5}$	$134 \pm 3 \times 10^{-5}$
M_3^4	$67 \pm 5 \times 10^{-5}$	$75 \pm 3 \times 10^{-5}$
$\beta^3(1917.1)$	$13.2° \pm 1.5°$	$18.4° \pm 0.7°$
g^3	$128 \pm 4 \times 10^{-6}\mathrm{d}^{-1}$	$118.9 \pm 0.6 \times 10^{-6}\mathrm{d}^{-1}$

The values of the M_j^μ are related to the eigenvectors of the coefficients' matrix in equation (5.15). If their numerical values given in this Chapter are used, we obtain, in units of the corresponding M_μ^μ, the set of values shown in Table 5.7.

Table 5.7. The Eigenvectors (units M_μ^μ)

μ	M_1^μ	M_2^μ	M_3^μ	M_4^μ
1	1	−0.0130	−0.0086	0
2	0.0057	1	−0.0421	−0.0001
3	0.0319	0.1677	1	−0.1060
4	0.0033	0.0172	0.0996	1

The values of the M_μ^μ are obtained from the observations (see Table 5.2) and are discussed in Chapter XI. A corrected set of values for the eigenvectors is obtained in Section 7.7 (see Table 7.3) when the Libration and its effects are taken into account.

6

The Equations of the Centre - II

6.1 The Forced Oscillations

After studying the free oscillations in longitude and radius vector we consider the calculation of a particular solution of the complete system of variational equations (5.5). The right hand sides are such that the solutions may be written as

$$\zeta_j = B_j \exp \mathrm{i}u + B'_j \exp \mathrm{i}u' \tag{6.1}$$

where the coefficients B_j and B'_j are undetermined. In order to obtain their approximate value, equations (6.1) are substituted into equations (5.5). Identifications of coefficients of the terms $\exp \mathrm{i}u$ and $\exp \mathrm{i}u'$ lead to two sets of four equations in B_j and B'_j. However the quasi-commensurable ratios n_1/n_2 and n_2/n_3 introduce significant corrections as shown in Chapter V and may not be disregarded. Here, they arise from the terms of second degree in the eccentricities whose arguments are $2u$ or $2u'$ given by the expression (3.28). To obtain the contribution of these terms, it must be kept in mind that the terms in (3.28) occur in R_{ij} exactly in the same way as in R_{ji} since the mutual distances do not depend on which satellite is disturbing or being disturbed. Using equations (4.4) and introducing these terms in equations (5.5), we have:

$$\begin{aligned}
\mathrm{i}\frac{d\zeta_1}{dt} &= \cdots - \left(b^1_{12}\zeta^*_1 - b^2_{12}\zeta^*_2\right) \exp 2\mathrm{i}u \\
\mathrm{i}\frac{d\zeta_2}{dt} &= \cdots - \left(b^2_{23}\zeta^*_2 - b^3_{23}\zeta^*_3\right) \exp 2\mathrm{i}u' + \left(b^1_{21}\zeta^*_1 - b^2_{21}\zeta^*_2\right) \exp 2\mathrm{i}u \\
\mathrm{i}\frac{d\zeta_3}{dt} &= \cdots + \left(b^2_{32}\zeta^*_2 - b^3_{32}\zeta^*_3\right) \exp 2\mathrm{i}u'
\end{aligned}$$

where

$$b_{ij}^{i} = \frac{Gm_j}{4n_i a_i^2}\left(44A_{ij}^4 + 14a_i\frac{\partial A_{ij}^4}{\partial a_i} + a_i^2\frac{\partial^2 A_{ij}^4}{\partial a_i^2}\right)$$
$$b_{ij}^{j} = \frac{Gm_j}{4n_i a_i^2}\left(42A_{ij}^3 + 14a_i\frac{\partial A_{ij}^3}{\partial a_i} + a_i^2\frac{\partial^2 A_{ij}^3}{\partial a_i^2}\right)$$
$$b_{ji}^{i} = \frac{Gm_i}{4n_j a_j^2}\left(42A_{ij}^3 + 14a_i\frac{\partial A_{ij}^3}{\partial a_i} + a_i^2\frac{\partial^2 A_{ij}^3}{\partial a_i^2}\right)$$
$$b_{ji}^{j} = \frac{Gm_i}{4n_j a_j^2}\left(38A_{ij}^2 + 14a_i\frac{\partial A_{ij}^2}{\partial a_i} + a_i^2\frac{\partial^2 A_{ij}^2}{\partial a_i^2}\right)$$

$j = i + 1$ and ζ^* is the complex conjugate of ζ. In order to get a particular solution of the complete system, we will introduce the angIe

$$\theta = u' - u. \tag{6.2}$$

In Section 5.4 we used the theorem of Laplace, after which $\theta = \pi$. In this Section, since the equations are part of the assumptions to demonstrate the theorem of Laplace, such procedure is not allowed. We will show that equation (6.1) is still an approximate particular solution of the complete system. Let the solutions of the complete system be written as

$$\zeta_j = \overline{B}_j \exp \mathrm{i}u \tag{6.3}$$

where

$$\overline{B}_j = \sum_{-\infty}^{+\infty} B_j^k \exp k\mathrm{i}\theta. \tag{6.4}$$

On multiplication by $\exp(-\mathrm{i}u)$ and substitution, it follows

$$\begin{aligned}
\mathfrak{m}\overline{B}_1 - \sum_k \{1,k\}\overline{B}_k &= -F_{12} - \left(b_{12}^1\overline{B}_1^* - b_{12}^2\overline{B}_2^*\right)\\
\mathfrak{m}\overline{B}_2 - \sum_k \{2,k\}\overline{B}_k &= -G_{12} - F_{23}e^{\mathrm{i}\theta} + \left(b_{21}^1\overline{B}_1^* - b_{21}^2\overline{B}_2^*\right) - \left(b_{23}^2\overline{B}_2^* - b_{23}^3\overline{B}_3^*\right)e^{2\mathrm{i}\theta}\\
\mathfrak{m}\overline{B}_3 - \sum_k \{3,k\}\overline{B}_k &= -G_{23}e^{\mathrm{i}\theta} + \left(b_{32}^2\overline{B}_2^* - b_{32}^3\overline{B}_3^*\right)e^{2\mathrm{i}\theta}\\
\mathfrak{m}\overline{B}_4 - \sum_k \{4,k\}\overline{B}_k &= 0.
\end{aligned}$$

In the calculation of the right hand side of each equation, we had a term $-\mathrm{i}\dot{\overline{B}}_j$, but we dropped this term; we assumed that they are very small when compared to the main coefficients. Also, in the forthcoming calculations, $\mathfrak{m}$ is taken as a constant. The numerical values of b_{ij}^k are

$$b_{12}^1 = 0.000\ 1915 \qquad b_{12}^2 = 0.000\ 2811$$

$$b_{21}^1 = 0.000\ 4089 \qquad b_{21}^2 = 0.000\ 5932$$
$$b_{23}^2 = 0.000\ 2879 \qquad b_{23}^3 = 0.000\ 4237$$
$$b_{32}^2 = 0.000\ 1095 \qquad b_{32}^3 = 0.000\ 1592.$$

The identification of the resulting equations in the powers of $\exp \mathrm{i}\theta$ leads to an infinite set of linear equations whose solutions for $|k| \leq 2$ are (units: 10^{-6})

$$\begin{aligned}
\overline{B}_1 &= 4333 + 136\ e^{-\mathrm{i}\theta} + e^{-2\mathrm{i}\theta} + 8\ e^{\mathrm{i}\theta} \\
\overline{B}_2 &= -2312 + 7412\ e^{\mathrm{i}\theta} + 49\ e^{2\mathrm{i}\theta} - 326\ e^{-\mathrm{i}\theta} - 2\ e^{-2\mathrm{i}\theta} \\
\overline{B}_3 &= -1 - 615\ e^{\mathrm{i}\theta} - 19\ e^{2\mathrm{i}\theta} \\
\overline{B}_4 &= -0.2\ e^{\mathrm{i}\theta}.
\end{aligned}$$

6.2 Induced Equations of the Centre

The effect of forced oscillations in the longitude and radius vector of the satellites is obtained by introducing the results of Section 6.1 in equations (4.5) and (4.6). We get

$$\delta r_i = -a_i \overline{B}_i \cos(\lambda_i - u_0)$$

$$\delta\theta_i = 2\overline{B}_i \sin(\lambda_i - u_0).$$

To calculate these inequalities, the Laplace theorem in the form $\theta = \pi$ was used again. Then

$$\begin{aligned}
\overline{B}_1 &= +0.004\ 190 \\
\overline{B}_2 &= -0.009\ 358 \\
\overline{B}_3 &= +0.000\ 598
\end{aligned}$$

and the inequalities are given by

$$\begin{aligned}
\delta r_1 &= -0.004190 a_1 \cos(\lambda_1 - u_0) & \delta\theta_1 &= +0.008380 \sin(\lambda_1 - u_0) \\
\delta r_2 &= +0.009358 a_2 \cos(\lambda_2 - u_0) & \delta\theta_2 &= -0.018716 \sin(\lambda_2 - u_0) \\
\delta r_3 &= -0.000598 a_3 \cos(\lambda_3 - u_0) & \delta\theta_3 &= +0.001197 \sin(\lambda_3 - u_0).
\end{aligned}$$

These inequalities are very similar to equations of the centre and on account of the fact that the motion of u_0 is very small ($-0.013\ \mathrm{d}^{-1}$), they have motions very close to the sidereal mean motions. They are called *induced equations of the centre* and the $|\overline{B}_i|$ are called *forced eccentricities*. According to Sampson the induced equations of the centre are

$$
\begin{aligned}
\delta r_1 &= -0.004124 a_1 \cos(\lambda_1 - u_0) \qquad & \delta\theta_1 &= \underline{+0.008230} \sin(\lambda_1 - u_0) \\
\delta r_2 &= +0.009430 a_2 \cos(\lambda_2 - u_0) \qquad & \delta\theta_2 &= \underline{-0.018678} \sin(\lambda_2 - u_0) \\
\delta r_3 &= -0.000633 a_3 \cos(\lambda_3 - u_0) \qquad & \delta\theta_3 &= +0.001204 \sin(\lambda_3 - u_0).
\end{aligned}
$$

In these equations, the two values underlined were deduced from observation and the remaining were calculated by Sampson. It is worthwhile to report de Sitter's values for the inequalities in longitude:

$$
\begin{aligned}
\delta\theta_1 &= \ +(8128 \pm 40) \times 10^{-6} \sin(\lambda_1 - u_0) \\
\delta\theta_2 &= \ -(18665 \pm 50) \times 10^{-6} \sin(\lambda_2 - u_0) \\
\delta\theta_3 &= \ +(1117 \pm 130) \times 10^{-6} \sin(\lambda_3 - u_0).
\end{aligned}
$$

To compare these results, we must remember that some inequalities having the same arguments as the induced equations of the centre have been calculated in Chapter IV (Tables 4.1 and 4.2) and adding these results to those obtained in this Section, we have

$$
\begin{aligned}
\delta r_1 &= -0.004169 a_1 \cos(\lambda_1 - u_0) \qquad & \delta\theta_1 &= +0.008312 \sin(\lambda_1 - u_0) \\
\delta r_2 &= +0.009367 a_2 \cos(\lambda_2 - u_0) \qquad & \delta\theta_2 &= -0.018533 \sin(\lambda_2 - u_0) \\
\delta r_3 &= -0.000638 a_3 \cos(\lambda_3 - u_0) \qquad & \delta\theta_3 &= +0.001210 \sin(\lambda_3 - u_0).
\end{aligned}
$$

These results agree well with Sampson's values.

6.3 Periodic Solutions of First Kind

In case of Jupiter I(Io) and Jupiter II(Europa), the induced equations of the centre have much more importance than the free inequalities. The induced inequalities have been observed since a long time and their amplitudes are well studied; prior to the space probes flight near Jupiter they have served in the determination of the physical parameters of the system (see for instance Section 11.6). For Jupiter III (Ganymede), the induced equation of the centre and the free equations have amplitudes of the same order of magnitude. One of the free equations served in the determination of the physical parameters (see Section 11.6).

The results of this Chapter show that the observed motions of the two inner satellites deviate from a uniform circular motion more because of the induced inequalities than because of the Keplerian elliptic inequalities. In view of this fact, de Sitter considered as starting point of his theory, a set of intermediary orbits already affected by these inequalities. It was first pointed out by Poincaré that the motion of the three inner Galilean satellites is very close to a periodic motion of first kind.

The periodic solutions of the first kind for k satellites orbiting around an oblate planet were studied by Griffin and by de Sitter for the three inner

Table 6.2. Forced Eccentricities in Periodic Solutions

Satellite	Griffin's Results Without Oblateness	Griffin's Results With Oblateness	de Sitter's Results	Section 6.2
I	0.0044	0.0026	0.00404	0.00417
II	0.0093	0.0086	0.00936	0.00937
III	0.0006	0.0006	0.00060	0.00064

Galilean satellites. For comparison we also show the relative coefficients of the radius vector inequalities calculated in Section 6.2.

The differences arise mainly from the set of values of the masses adopted by Griffin; he considered Laplace's old determination, which, in some cases, is wrong by a factor 2. The results shown in the last column include the effect of Jupiter IV(Callisto) and Sun.

6.4 Osculating Eccentricities and Perijoves

The osculating eccentricities and longitudes of perijoves are composed in a single complex variable

$$\zeta_i = e_i \exp \mathrm{i}\varpi_i$$

which yields $e_i = |\zeta_i|$ and $\varpi_i = \arg \zeta_i$. For the first two satellites, ζ_i does not differ too much from the forced oscillations. The amplitudes of the free oscillations in this case are only a few percent of the amplitudes of the forced oscillations. The osculating values oscillate slightly around the values

$$e_1 = 0.00419 \qquad \varpi_1 = u$$

$$e_2 = 0.00936 \qquad \varpi_2 = \pi + u.$$

In the case of Jupiter IV (Callisto), there are no forced oscillations like the induced equations of the centre and the three composed free oscillations are much more smaller than the main one. Thus, the osculating values oscillate slightly around the values

$$e_4 = M_4^4 (= 0.00733) \qquad \varpi_4 = g^4 t + \beta^4$$

In the case of Jupiter III (Ganymede), the situation is quite different from others and we have to consider at least two free oscillations added to the forced oscillation $\overline{B}_3 \exp \mathrm{i}u$. Using observed values for the free oscillations, we obtain

$$\zeta_3 = 0.00147 \exp \mathrm{i}(g^3 t + \beta^3) + 0.00064 \exp \mathrm{i}(g^4 t + \beta^4) + 0.00060 \exp \mathrm{i}u.$$

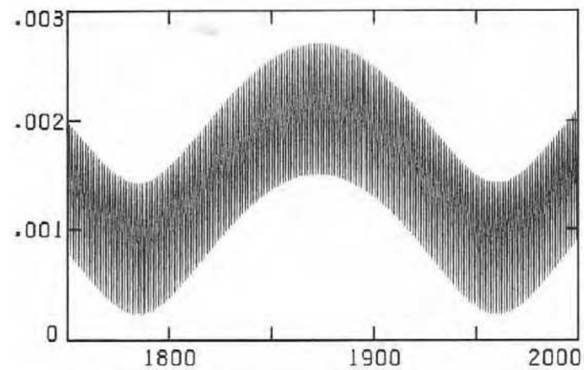

Fig. 6.1. Eccentricity of J3 (Ganymede)

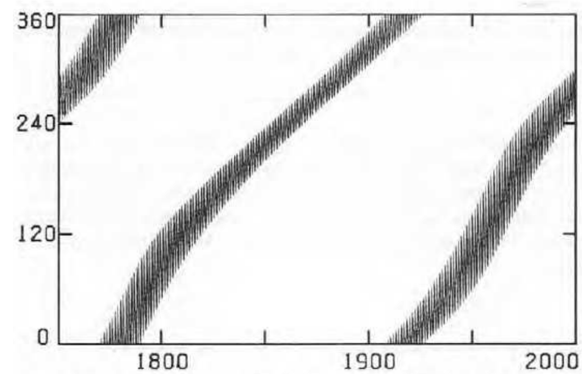

Fig. 6.2. Perijove of J3 (Ganymede)

Terms below 10^{-5} have been neglected. One consequence of this relation is that $|\zeta_3|$ may never become equal to zero. The minimum value that the osculating eccentricity can attain is ~ 0.0002. Since u is much faster than the proper perijoves, the eccentricity passes from relative minima to relative maxima (and vice-versa) in 8 months. The free oscillations modulate this rhytm: the deeper minima are reached at every 180 years and have occurred around 1961. This periodic behavior of the free oscillations is the same considered by Laplace and has been told in Section 5.6.

The motion of the osculating perijove of Jupiter III is dominated by the motion of the proper perijove of the satellite. One oscillation of total amplitude 50 degrees and period 180 years due to the other free oscillation is superimposed. The 16-month component due to the forced inequality has variable amplitude: it is 86 degrees at the epoch of the deepper minima of the osculating eccentricity and it is only 32 degrees at the epoch of the higher maxima of e_3.

References and Notes

- 6.3

  See

  F.L.Griffin: 1920, "Certain Periodic Orbits of k Finite Bodies revolving about a relatively Large Central Mass", in F.R.Moulton, *Periodic Orbits*. Carnegie Institution, Washington, 425-456.

  W. de Sitter: 1909, "On the Periodic Solutions of a Special Case of the Problem of Four Bodies", *Proc. Acad. Amsterdam, 11*, 682-698.

- 6.4

  We have adopted the proper perijove values given in J.Lieske's Ephemeris E-2 (see Section 11.8). The eigenvalues and forced eccentricities are those from this book. The results agree well with recent numerical integrations of the averaged equation done by M. Sato in a joint project directed by P.Nacozy and the author. A numerical integration of averaged equations has been also performed by B.Brown, but he used a different set of integration constants.

  B.C.Brown: 1976, "The Orbits of the Galilean Satellites of Jupiter", Ph.D.Thesis, University of Maryland, College Park.

  The results obtained for ϖ_1 and ϖ_2 show that the angles $x_1 = u - \varpi_1$, $x_2 = u - \varpi_2$ and $x_3 = u' - \varpi_2$ behave like librators around $x_1 = 0$, $x_2 = \pi$ and $x_3 = 0$. Sinclair called this phenomenon *short-period librations.*

  A.T.Sinclair: 1975, " The Orbital Resonance amongst the Galilean Satellites of Jupiter". *Celestial Mechanics, 12*, 89-96.

  The angle $x_4 = u' - \varpi_3$ circulates. It is noteworthy that the periodic solutions considered in Section 6.3 correspond to $x_1 = x_3 = 0$, $x_2 = x_4 = \pi$.

7

The Libration

7.1 The Laplacian Theory

In Section (5.4), the equations for the mean longitudes were considered and the great inequalities arising out of the free equations of the centre were calculated. The indirect effect arising out of the induced equations of the centre, discovered by Laplace, is a completely different phenomenon. Let the forced terms (6.3) be substituted in equation (5.11). 1t then follows

$$\begin{aligned} \frac{d^2\rho_1}{dt^2} &= -\sum_k K_{12}^k \sin k\theta \\ \frac{d^2\rho_2}{dt^2} &= -\sum_k \left[K_{23}^k \sin(k-1)\theta + L_{12}^k \sin k\theta\right] \\ \frac{d^2\rho_3}{dt^2} &= -\sum_k L_{23}^k \sin(k-1)\theta \end{aligned} \tag{7.1}$$

where

$$K_{ij}^k = -3n_i F_{ij} B_i^k - 3n_j \frac{m_j a_j^2}{m_i a_i^2} G'_{ij} B_j^k$$

$$L_{ij}^k = 6n_j G_{ij} B_j^k + 6n_i \frac{m_i a_i^2}{m_j a_j^2} F_{ij} B_i^k .$$

This is not the only instance where the arguments $k\theta$ appear. In fact if the induced equations of the centre are substituted in the equations for $\varepsilon_i^{\mathrm{I}}$ corresponding to the same parts of the disturbing function considered in equations (7.1), and if the resulting equations are differentiated with respect to time we obtain equations like

$$\frac{d^2\varepsilon_1^{\mathrm{I}}}{dt^2} = -\sum \Delta K_{12}^k \sin k\theta ,$$

etc. The coefficients ΔK_{12}^k, etc., are just a few percent of the corresponding coefficients in equations (7.1), and in a first approximation its contribution needs not to be taken into account.

In each case, the coefficients are at least quadratic in the satellite masses. Many similar terms exist when all contributions of the second degree in the masses are considered. In Sampson's theory, there are 25 terms contributing to each term K_{12}^1, and K_{12}^2, and so on. While the terms considered here are the most important, the remaining, being large in number, in total give a non-negligeable contribution.

In this book we are just interested in the characteristic of the phenomenon. We will limit the equations to their main part as it has been done by Laplace. Let us consider just the contributions arising from the main B_j^k and use the approximations

$$B_1^0 = \frac{-F_{12}}{\mathfrak{m} + g_1} \qquad B_2^0 = \frac{-G_{12}}{\mathfrak{m} + g_1}$$
$$B_2^1 = \frac{-F_{23}}{\mathfrak{m} + g_1} \qquad B_3^1 = \frac{-G_{23}}{\mathfrak{m} + g_1}$$

and $G'_{ij} = G_{ij}$ (since $a_j^3/a_i^3 \simeq 4$). Thus

$$\begin{aligned} \frac{d^2\rho_1}{dt^2} &= K\frac{m_2 m_3}{a_1^2}\sin\theta \\ \frac{d^2\rho_2}{dt^2} &= -3K\frac{m_1 m_3}{a_2^2}\sin\theta \\ \frac{d^2\rho_3}{dt^2} &= 2K\frac{m_1 m_2}{a_3^2}\sin\theta \end{aligned} \tag{7.2}$$

where

$$K = -\frac{3n_2 a_2^2}{\mathfrak{m} + g_1}\frac{G_{12}F_{23}}{m_1 m_3}.$$

The way of integration of equations (7.2) is different from that of equations (5.13), since there θ was supposed to be a constant (π). The differential equation in θ is

$$\frac{d^2\theta}{dt^2} = \frac{d^2}{dt^2}(\lambda_1 - 3\lambda_2 + 2\lambda_3).$$

As we have neglected the effects arising from $\varepsilon_i^{\mathrm{I}}$, we have to consider just

$$\frac{d^2\theta}{dt^2} = \frac{d^2}{dt^2}(\rho_1 - 3\rho_2 + 2\rho_3)$$

which results

$$\frac{d^2\theta}{dt^2} = C_1 \sin\theta \tag{7.3}$$

where

$$C_1 = K\left(\frac{m_2m_3}{a_1^2} + \frac{9m_1m_3}{a_2^2} + \frac{4m_1m_2}{a_3^2}\right) \tag{7.4}$$

is the Laplacian approximation for coefficient C_1. From the figures in preceding Chapters we have $K = 2.87 \times 10^4 \mathrm{b}_0^2\mathrm{d}^{-2}$ and $C_1 = 1.30 \times 10^{-5}\mathrm{d}^{-2}$

In the discussion of the consequences of the libration, it is convenient to introduce the *factors of libration*

$$Q_1 = \frac{Km_2m_3}{C_1a_1^2} \qquad Q_2 = -\frac{3Km_1m_3}{C_1a_2^2} \qquad Q_3 = \frac{2Km_1m_2}{C_1a_3^2}$$

which are the coefficients of $\sin\theta$ in equations (7.2) in units of C_1. This simplified calculation leads to

$$Q_1 = 0.127 \qquad Q_2 = -0.275 \qquad Q_3 = 0.024.$$

It is noteworthy that

$$Q_1 - 3Q_2 + 2Q_3 = 1.$$

7.2 Possible Solutions of Equation (7.3)

Equation (7.3) is the equation of the simple pendulum and may be solved exactly in terms of elliptic functions. Multiplication of it by $d\theta/dt$ yields an exact differential equation which on integration leads to

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = C_0 - C_1\cos\theta \tag{7.5}$$

where C_0 is an integration constant. It then follows

$$dt = (2C_0 - 2C_1\cos\theta)^{-1/2}d\theta$$

and the elliptic integral

$$t = t_0 + \int_{\theta_0}^{\theta}(2C_0 - 2C_1\cos\theta)^{-1/2}d\theta.$$

If $C_0 > C_1$, the quantity under the radical sign is positive for all θ and the function $t(\theta)$ defined by the integral is monotonic: the solutions are *circulations*. The angle θ circulates and the period of the circulation is given by

$$T = \int_0^{2\pi}(2C_0 - 2C_1\cos\theta)^{-1/2}d\theta.$$

The time spent in going from $\theta = \pi/2$ to $\theta = 3\pi/2$ is

$$T_1 = \int_{\pi/2}^{3\pi/2}(2C_0 - 2C_1\cos\theta)^{-1/2}d\theta.$$

In the integration interval, $\cos\theta \leq 0$, thus

$$T_1 < \frac{\pi}{\sqrt{2C_1}}$$

that is $T_1 < 616$ days. This variation in θ would be greater than 0.3 degrees per day and would not escape detection. On contrary, observations show that θ is almost constant and equal to π.

If $C_0 = C_1$, we may perform the integration by means of elementary functions (this integral is also called lambda function)

$$t = t(\pi) + \frac{1}{\sqrt{C_1}} \log|\tan\frac{\theta}{4}|.$$

It is noteworthy that the integral diverges at the boundaries $\theta = 0 (\bmod 2\pi)$. In this separatrix solution the angle θ has been in the remote past close to the unstable equilibrium point $\theta = 0$ and is going to be close again of this same point, in the remote future, after performing one complete revolution (in an infinite time). Under such condition, the angular velocity

$$\frac{d\theta}{dt} = 2\sqrt{C_1}\sin\frac{\theta}{2}$$

reaches its maximum value $2\sqrt{C_1}$ when $\theta = \pi$ (0.4 degrees per day), and this solution may be discarded. If θ is close to π, as observed, the solution is not almost constant whereas if it is constant θ should be close to 0.

This fact leads to $C_0 < C_1$. In order to discuss these solutions we introduce θ_0 defined by the relation

$$C_0 = C_1 \cos\theta_0 \qquad\qquad (0 \leq \theta_o < \pi).$$

Then we get,

$$t = t_0 + \int_{\theta_0}^{\theta} \left[2C_1(\cos\theta_0 - \cos\theta)\right]^{-1/2} d\theta. \tag{7.6}$$

To have a real solution, $\cos\theta < \cos\theta_0$, that is, $\theta_0 < \theta < 2\pi - \theta_0$. Let it be remarked from the Jacobian integral (7.5) that $d\theta/dt$ is equal to zero only at the boundaries θ_0 and $2\pi - \theta_0$. If $d\theta/dt$ is positive at the initial time, the angle θ will increase from its initial value to the boundary $2\pi - \theta_0$; at this point, the angular velocity $d\theta/dt$ becomes zero and changes its sign. The angle θ will start decreasing and will reach the boundary θ_0. Once more $d\theta/dt$ becomes zero, changes its sign and the evolution of θ is changed. The solutions in that case are called *librations*. The angle θ oscillates between the boundaries θ_0 and $2\pi - \theta_0$. The amplitude of the oscillation $(2\pi - 2\theta_0)$ depends on the integration constant C_0. The period of libration also depends on the integration constant and is given by

$$T = 4\int_{\theta_0}^{\pi} (2C_0 - 2C_1\cos\theta)^{-1/2} d\theta. \tag{7.7}$$

7.3 The Libration

Let the auxiliary angle Φ be defined by the standard transformation

$$\sin \Phi = -\frac{1}{k} \cos \frac{\theta}{2}$$

where

$$k^2 = \frac{C_0 + C_1}{2C_1}.$$

Equation (7.6) then becomes

$$t = t_0 + \frac{1}{\sqrt{C_1}} \left[F(\Phi, k) - F(\Phi_0, k) \right]$$

where $F(\Phi, k)$ stands for the elliptic integral of the first kind. If the starting point is the stable equilibrium position $\theta_0 = \pi$, we have

$$t = t(\pi) + \frac{1}{\sqrt{C_1}} F(\Phi, k).$$

The inversion may be performed by means of the Jacobian elliptic functions:

$$\cos \frac{\theta}{2} = -k. \sin \Phi = -k.\mathrm{sn}[\sqrt{C_1}(t - t(\pi))].$$

The period of the libration already given in equation (7.7) becomes

$$T = \frac{4}{\sqrt{C_1}} K(k) \tag{7.8}$$

where $K(k)$ stands for the complete elliptic integral:

$$K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \Phi)^{-1/2} d\Phi.$$

$K(k)$ is a monotonic function and for $0 < k < 1$ we have $\pi/2 < K(k) < \infty$.

The amplitude and period of the libration depend on the value of the integration constant C_0 and must be determined from observations. This determination is very difficult because the amplitude of the libration is very small and its period is not accurately calculated. In 1907, de Sitter, assuming the value of the period, found 0.16 ± 0.05 degrees for the half-amplitude. The standard error is underestimated. In 1928, in a new discussion, the amplitude, period and phase were determined; he then found 0.025 ± 0.01 degrees for the half-amplitude and 2180 ± 60 days for the period. A separate determination using only Jupiter I(Io) gave nearly the same phase but an amplitude about four times larger, while using only Jupiter II(Europa) he obtained the same amplitude but a difference of 100 degrees in phase. The determination is thus extremely uncertain and the conclusion is that libration is too small to be detected from old observations. It is noteworthy that Brouwer in the determination of libration from the Johannesburg series of eclipse observations found practically the same values for the half-amplitude (0.031 ± 0.009 degrees) and phase assuming the period equal to 2050 days. However the separate analysis of every satellite gave contradictory results and thus he concluded that the determination only showed the libration to be too small to be determined from the observations.

The agreement of two absolutely independent determinations from different observations and by different investigators is, however, very remarkable.

Recent determinations by Lieske show amplitudes twice greater (see Section 11.8). These results show that the amplitude of the libration around the stable equilibrium point π is very small. We may write

$$\frac{d^2\theta}{dt^2} = C_1(\pi - \theta) \tag{7.9}$$

instead of equation (7.3). The solution of equation (7.9) is an harmonic oscillation around the center of libration π:

$$\theta = \pi + D\sin(n_L t + E) \tag{7.10}$$

where $n_L = \sqrt{C_1}$; D and E are integration constants. We have, in this approximation,

$$\sin\theta = D\sin(n_L t + E)$$

and equations (7.2) become

$$\frac{d^2\rho_1}{dt^2} = \frac{m_2 m_3}{a_1^2} KD\sin(n_L t + E),$$

etc. On integration, we get the following inequalities, called *Librations* by Laplace:

$$\begin{aligned} \delta\theta_1 &= -Q_1 D\sin(n_L t + E) \\ \delta\theta_2 &= -Q_2 D\sin(n_L t + E) \\ \delta\theta_3 &= -Q_3 D\sin(n_L t + E). \end{aligned} \tag{7.11}$$

7.4 The Period of Libration

The period of libration is $2\pi/n_L$ and was first determined to be equal to 2270 days by Laplace using the equations of Section 7.3. The values of the masses used by Laplace in some cases were wrong by a factor more than two. When one uses the values of the masses adopted by the International Astronomical Union, one obtains 1740 days. An accurate determination was made by Sampson who considered the entire set of terms whose argument was $k\theta$ and, for the coefficients of $\sin\theta$ and $\sin 2\theta$, he found

$$C_1 = 11.75 \times 10^{-6}\mathrm{d}^{-2} \qquad C_2 = 1.14 \times 10^{-6}\mathrm{d}^{-2},$$

respectively. For small librations, the solution is still given by equation (7.10), where

$$n_L = \sqrt{C_1 - 2C_2}$$

and the correaponding period is 2042 days. If this period is recalculated using the present adopted masses, it will become smaller by about 30 days.

The results of Sampson have been revitalized in the past years at the Jet Propulsion Laboratory (Pasadena, Cal.) and at the Bureau des Longitudes (Paris). At the JPL, Lieske obtained 2094 days and at the BdL, Vu obtained 2240 days. In each case the Sampson set of masses was adopted. Using a set of masses close to the set used in this text (within 4%), Brown obtained 2032 days.

The main difference between these results and Laplace's results is that in all theories developed in this century the terms of second degree in the equations of perturbations have been fully considered. The importance of the inclusion of second degree terms in the solution of libration has explicite1y been pointed out by Marsden in 1966.

7.5 Laplace Theorems

The results of Section 7.3 may be summarized in the two theorems stated by Laplace:

(a) The time-average of $n_1 - 3n_2 + 2n_3$ is zero; and
(b) The time-average of $\varepsilon_1 - 3\varepsilon_2 + 2\varepsilon_3$ is π.

In fact, the libration is very small and we may write $n_1 - 3n_2 + 2n_3 = 0$ and $\varepsilon_1 - 3\varepsilon_2 + 2\varepsilon_3 = \pi$. There are some interesting kinematical consequences of these theorems. The three satellites may not have a triple conjunction, that is, a situation in which the three satellites are on the same side of the planet and on a straight line with Jupiter. In fact, the three situations in which a *conjunction* happens are as follows:

(a) If Jupiter I(Io) and Jupiter II(Europa) are in conjunction, that is, $\lambda_1 = \lambda_2 \pmod{2\pi}$, then, necessarily, $\lambda_2 - \lambda_3 = \pi/2 \pmod{\pi}$. This shows that the radius vector of Jupiter III (Ganymede) is perpendicular to the line of conjunction of the two inner satellites. The situation of Ganymede relative to Io and Europa is a *quadrature*.

(b) If Europa and Ganymede are in conjunction, that is $\lambda_2 = \lambda_3 \pmod{2\pi}$, then $\lambda_1 - \lambda_2 = \pi \pmod{2\pi}$; this means that Io lies in the same straight line as the other two and the planet, but on the opposite side of Jupiter. The situation of Io relative to Europa and Ganymede is an *opposition*.

(c) If Io and Ganymede are in conjunction, that is, $\lambda_1 = \lambda_3 \pmod{2\pi}$, then $\lambda_1 - \lambda_2 = \pi \pmod{2\pi/3}$; Europa is in opposition to the other satellites or it is in the same side of the planet as the other two but its radius vector is 60 degrees away of the conjunction line.

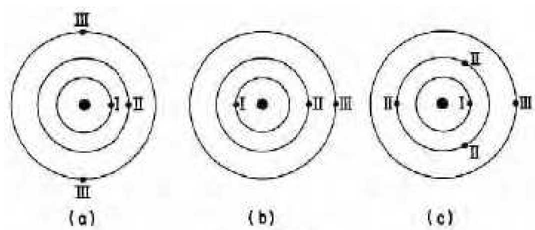

Fig. 7.1.

It may be kept in mind that these results refer to the mean satellite. In fact the inequalities in longitude studied in earlier chapters may move the satellite forward or backward from their mean positions by as much as one arc degree.

7.6 Indirect Effects

Let some long period inequalities in the mean longitude be written

$$\begin{aligned} \delta\lambda_1 &= L_1 \sin(\alpha t + \beta) \\ \delta\lambda_2 &= L_2 \sin(\alpha t + \beta) \\ \delta\lambda_3 &= L_3 \sin(\alpha t + \beta). \end{aligned} \tag{7.12}$$

It does not matter how they arose; let it just be assumed that they were calculated in an identical way as inequalities in the mean longitude were calculated earlier. The corresponding second-order differential equations are

$$\frac{d^2\lambda_i}{dt^2} = -L_i \alpha^2 \sin(\alpha t + \beta).$$

If we do not integrate as usual but add the right-hand side of these equations to those of equations (7.2), we obtain

$$\frac{d^2\lambda_i}{dt^2} = Q_i C_i \sin\theta - \alpha^2 L_i \sin(\alpha t + \beta) \tag{7.13}$$

and, instead of equation (7.9), we have

$$\frac{d^2\theta}{dt^2} = C_1(\pi - \theta) - \alpha^2 L \sin(\alpha t + \beta)$$

where

$$L = L_1 - 3L_2 + 2L_3.$$

Thus, to the general solution (7.10), we have to add the particular solution

$$\theta = \frac{\alpha^2 L}{C_1 - \alpha^2} \sin(\alpha t + \beta).$$

When this additional term is introduced in equations (7.13), the integration results, besides libration, the inequalities

$$\delta\lambda_i = \left(L_i + Q_i L \frac{C_1}{\alpha^2 - C_1}\right) \sin(\alpha t + \beta). \tag{7.14}$$

If these results are compared with former values of the inequalities (7.12), we find that the coefficients of the inequalities were modified and the modifications are as important as the period of the inequalities approaches the period of the librations.

Let the results of this section be applied to the Great Inequalities in the mean longitudes introduced in Section 5.4. Equations (5.14) were obtained by direct integration of equations (5.13) and the numerical results are shown in Table 7.1. Table 7.1 also shows the results obtained after correction of the libration effects by means of equation (7.14). In general the corrections are not great since the periods of the Great Inequalities are not close enough of the period of libration. However, in the greatest among these inequalities,

$$\delta\rho_2 = 4'03'' \sin(u - g^3 t - \beta^3), \tag{7.15}$$

Table 7.1. Great Inequalities in the Mean Longitudes

Satellite	Argument	Factors multiplying the Forced Eccentricities: Sampson	de Sitter	Brown	Equations (5.14)	Libration Corrected	Coefficients
	u-g^1t-β^1	−2.704	−2.705	−3.04	−2.95	−3.03	−0.00003
I	u-g^2t-β^2	+0.833	+0.925	+1.34	+1.37	+1.36	+0.00012
Io	u-g^3t-β^3	+0.104	+0.094	+0.134	+0.123	+0.136	+0.00020
	u-g^4t-β^4	+0.0084	+0.0155	+0.0123	+0.0127	+0.0140	+0.00010
	u-g^1t-β^1	+4.288	+3.888	+4.41	+4.22	+4.44	+0.00004
II	u-g^2t-β^2	+1.619	+1.635	+1.18	+1.01	+1.06	+0.00010
Europa	u-g^3t-β^3	−0.513	−0.584	−0.850	−0.767	−0.804	−0.00118
	u-g^4t-β^4	−0.0426	−0.1055	−0.0734	−0.0759	−0.0801	−0.00059
	u-g^1t-β^1	−0.067	−0.030	0	+0.01	−0.01	–
III	u-g^2t-β^2	−0.702	−0.717	−0.762	−0.76	−0.77	−0.00007
Ganym.	u-g^3t-β^3	+0.077	+0.121	+0.164	+0.142	+0.145	+0.00021
	u-g^4t-β^4	+0.0069	+0.0219	+0.0126	+0.0139	+0.0142	+0.00010

the correction reaches 11″. This correction is to be considered big if we remember that in Sampson's tables of the Galilean satellites all effects greater than 0.02″ were supposed to be considered. Nevertheless the most striking features in Table 7.1 are the great differences between corresponding values; they indicate that the results are very sensitive to the values of the masses and also to the perturbations technique adopted in the calculations. Thus, in Sampson's tables, the coefficient of the inequality corresponding to equation (7.15) is only 2′36″.

7.7 Effects on the Free Oscillations

If the inequalities that appear in the right-hand side of equation (5.15) are affected by libration then the coefficients a_{jk} in equations (5.16) are also affected. As a consequence, the characteristic roots g^μ and the eigenvectors of the coefficients' matrix are also affected . The values of the characteristic roots obtained by equation (5.16) with corrected a_{jk} values are shown in Table 7.2, where, in addition, values obtained by Sampson, de Sitter and Lieske are also included.

Table 7.2. The Characteristic Roots (units: $10^{-6}\mathrm{d}^{-1}$)

μ	Eqn.(5.16)	Sampson	de Sitter	Lieske
1	2731	2756		2810
2	700	822	<u>700</u>±39	814
3	130	121	<u>123</u>±5	124
4	32.0	<u>32.4</u>	<u>32.9</u>± 0.5	32.1

Note: The values underlined are deduced from the observations

The corresponding values of the M_j^μ are shown in Table 7.3. These values may be compared to the set of values recently obtained by Brown (Table 7.4) in a complete second-order calculation.

Table 7.3. The Eigenvectors (units: M_μ^μ)

μ	M_1^μ	M_2^μ	M_3^μ	M_4^μ
1	1	-0.0115	-0.0089	0
2	0.0039	1	-0.0424	-0.0001
3	0.0331	0.1690	1	-0.1059
4	0.0034	0.0173	0.0994	1

The discrepancies in the fourth column of Tables 7.3 and 7.4 are the most important since the proper eccentricity of Jupiter IV (Callisto) M_4^4 is much

Table 7.4. Brown's Eigenvectors (units: M_μ^μ)

μ	M_1^μ	M_2^μ	M_3^μ	M_4^μ
1	1	-0.0039	-0.0082	0
2	-0.0016	1	-0.0452	-0.0001
3	0.0351	0.1686	1	-0.1127
4	0.0031	0.0152	0.0882	1

greater than the eccentricities of the inner satellites. A simple analysis of the formation of the M_k^4 ($k \neq 4$) shows that they are given by a linear combination of the $\{j,4\}$ ($j \neq 4$). The coefficients in this combination depend weakly on m_4 (the dependence is the same as of g^4 on m_4) while the $\{j,4\}$ ($j \neq 4$) are proportional to m_4. Brown has used a value of m_4 close to Sampson's and we used the value derived at the Jet Propulsion Laboratory from the analysis of the orbit of space probes Pioneer 10 and 11 and recommended by the International Astronomical Union. For comparison with Brown's results our values of M_j^4 ($j \neq 4$) must be reduced by 16 percent.

The most important remark is related to M_3^4. We obtained $M_3^4 = 0.0994 M_4^4$. Since the observed value of M_4^4 is 0.0073, it follows $M_3^4 = 0.00072$. Sampson and de Sitter, from observations, independently obtained 0.00064 and 0.00067 (with standard error 0.00004) for M_3^4. Our value for M_3^4 is not consistent with the observational values. Future research must decide on three alternatives: (1) Sampson's and de Sitter's values for M_3^4 with different observations are too small, (2) the value of m_4 obtained from the path analysis of Pioneer 10 and 11 is too high, and (3) it is not possible to relate M_3^4 with m_4 accurately in current theories. It is noteworthy that one determination by de Sitter using old and modern observations gave 0.00075 (see Section 5.6), which is the expected value.

7.8 Effects of Quadratic Inequalities

We may also evaluate what would happen in the presence of actions, like dissipative forces, that would lead to quadratic inequalities in the mean longitudes. For instance, let these inequalities be calculated as $L_1 t^2$, $L_2 t^2$ and $L_3 t^2$ when libration is completely disregarded. The corresponding second order differential equations are

$$\frac{d^2\lambda_i}{dt^2} = 2L_i \qquad\qquad (i = 1, 2, 3).$$

Because of the libration, these actions should be considered together with equations (7.2); they contribute the additive quantity $2L_i$ to the right-hand side of each one of equations (7.2). In equation (7.9), we obtain the additional forced term $2L$ where $L = L_1 - 3L_2 + 2L_3$. Thus, we have

$$\frac{d^2\theta}{dt^2} = C_1(\pi - \theta) + 2L$$

whose general solution is

$$\theta = \pi + \frac{2L}{C_1} + D\sin(n_L t + E). \tag{7.16}$$

If this solution is introduced in the composed second-order equations, we obtain, besides librations, the inequalities

$$\delta\lambda_1 = (L_1 - Q_1 L)t^2$$
$$\delta\lambda_2 = (L_2 - Q_2 L)t^2$$
$$\delta\lambda_3 = (L_3 - Q_3 L)t^2.$$

This set of solutions contains two noteworthy facts: (i) the libration center deviates from π to $\pi+2L/C_1$, and (ii) the individual effects due to the physical agent on each satellite are redistributed among themselves in such a way that the Laplacian relation $n_1 - 3n_2 + 2n_3 = 0$ is preserved.

This shows a reality in the dynamical evolution of the Galilean system. As an example, let us suppose that a resisting medium exists around Jupiter and reaches the orbit of Jupiter I(Io) but not the outermost orbits. If Io was orbiting alone this fictitious condition would give rise to an acceleration Lt^2 in its longitude. Because of the resonance this effect becomes smaller and is partly redistributed to the outer satellites. The calculations yield

$$\delta\lambda_1 \simeq 0.873Lt^2, \quad \delta\lambda_2 \simeq 0.275Lt^2, \quad \delta\lambda_3 \simeq -0.024Lt^2.$$

Note that the acceleration of the third satellite would be negative and also that $\dot{\mathfrak{m}} = 0.33L$ (positive). On contrary, if tidal torques are acting on the satellites, L is negative along with $\dot{\mathfrak{m}}$ and the relation $n_1 - 2n_2$ tends to become more close to zero than it is now.

References and Notes

- 7.1
  The libration factors given in this Section may be compared to values derived from other works.

Table 7.5. Libration Factors

	Q_1	Q_2	Q_3
Section 7.1	0.127	−0.275	0.024
Sampson Tables	0.1381	−0.2704	0.0254
Lieske's Ephemeris E-2	0.1152	−0.2767	0.0274

- 7.2
  It is worthwhile to mention that a Laplacian resonance also happens among the inner satellites of Uranus whose mean motions are

Uranus V (Miranda):	$n_5 = 4.4452\ \mathrm{d}^{-1}$
Uranus I (Ariel):	$n_1 = 2.4933\ \mathrm{d}^{-1}$
Uranus II (Umbriel):	$n_2 = 1.5162\ \mathrm{d}^{-1}$.

  Then
  $$n_5 - 3n_1 + 2n_2 = -0.0024\ \mathrm{d}^{-1}.$$
  Nevertheless, in this case, the commensurability is not close enough to get a libration. The angle θ circulates. The theory presented in this book does not apply to Uranus satellites since, in that case, $\mathfrak{m}$ is not a small quantity ($\mathfrak{m} = -0.54\ \mathrm{d}^{-1}$) and the forced terms in equations (5.5) and (5.11) may not be restricted to those whose arguments are u and u'.

- 7.4
  With the present adopted masses Lieske's result for the period of libration becomes 2074 days.

- 7.6
  de Sitter results in Table 7.1 are from
  W. de Sitter: 1928, "Orbital Elements determining the longitudes of Jupiter' s Satellites derived from Observations", *Annalen Sterrewacht Leiden* XVI(2).
  The coefficients in the last column were obtained using the proper eccentricities of Lieske's ephemeris E-2 (see Section 11.8).

8

Solar Effects

8.1 The Variations

The disturbing solar forces derive from the force-functions R_{i0} calculated in Section 3.2 (eqns. 3.5). The resulting inequalities are similar to those found in the theory of the Moon and they are named after the classical denominations of the lunar theory. We call *variations* the inequalities whose arguments are $2\lambda_i - 2\lambda_0$. They arise from

$$\frac{3}{4}\frac{Gm_0}{a_0}\left(\frac{a_i}{a_0}\right)^2 \big(\cos(2\lambda_i - 2\lambda_0) + e_i\cos(2\lambda_0 - 3\lambda_i + \varpi_i) - 3e_i\cos(2\lambda_0 - \lambda_i - \varpi_i)\big).$$

The corresponding variational equations are

$$\begin{aligned}
\frac{da_i}{dt} &= -3\frac{Gm_0 a_i}{n_i a_0^3}\sin(2\lambda_i - 2\lambda_0)\\
\frac{d^2\rho_i}{dt^2} &= \frac{9}{2}\frac{Gm_0}{a_0^3}\sin(2\lambda_i - 2\lambda_0)\\
\frac{d\varepsilon_i^{\mathrm{I}}}{dt} &= -3\frac{Gm_0}{n_i a_0^3}\cos(2\lambda_i - 2\lambda_0)\\
\frac{d\zeta_i}{dt} &= \frac{3}{4}\frac{Gm_0\mathfrak{i}}{n_i a_0^3}\big(\exp\mathfrak{i}(-2\lambda_0 + 3\lambda_i) - 3\exp\mathfrak{i}(2\lambda_0 - \lambda_i)\big)
\end{aligned}$$

except for terms that are of the order of the eccentricities. The integration leads to

$$\begin{aligned}
\delta a_i &= \frac{3Gm_0 a_i}{2n_i a_0^3(n_i - n_0)}\cos(2\lambda_i - 2\lambda_0)\\
\delta\rho_i &= -\frac{9Gm_0}{8a_0^3(n_i - n_0)^2}\sin(2\lambda_i - 2\lambda_0)
\end{aligned}$$

$$\begin{aligned}
\delta\varepsilon_i^{\mathrm{I}} &= -\frac{3Gm_0}{2n_i a_0^3(n_i - n_0)}\sin(2\lambda_i - 2\lambda_0)\\
\delta\zeta_i &= \frac{3Gm_0}{4n_i a_0^3}\left(\frac{\exp\mathrm{i}(3\lambda_i - 2\lambda_0)}{3n_i - 2n_0} - \frac{3\exp\mathrm{i}(2\lambda_0 - \lambda_i)}{2n_0 - n_i}\right).
\end{aligned}$$

The corresponding inequalities in radius vector and longitude are obtained when these results are used in equations (4.5) and (4.6), It then follows

$$\begin{aligned}
\delta r_i &= \frac{3Gm_0 a_i}{4n_i a_0^3}\left(\frac{2}{n_i - n_0} - \frac{1}{3n_i - 2n_0} - \frac{3}{n_i - 2n_0}\right)\cos(2\lambda_i - 2\lambda_0)\\
\delta\theta_i &= -\frac{3Gm_0}{4n_i a_0^3}\left(\frac{7n_i - 4n_0}{2(n_i - n_0)^2} + \frac{2}{3n_i - 2n_0} - \frac{6}{n_i - 2n_0}\right)\sin(2\lambda_i - 2\lambda_0).
\end{aligned}$$

An important simplification follows immediately since $n_i \gg n_0$ and n_0 may be neglected in all combinations with n_i. Thus, we obtain simplified equations:

$$\begin{aligned}
\delta r_i &= -\frac{Gm_0 a_i}{n_i^2 a_0^3}\cos(2\lambda_i - 2\lambda_0)\\
\delta\theta_i &= \frac{11}{8}\frac{Gm_0}{n_i^2 a_0^3}\sin(2\lambda_i - 2\lambda_0).
\end{aligned}$$

To obtain the usual simplified equations of the variation, we introduce $Gm_0 = n_0^2 a_0^3$ (m_0 is the mass of the Sun) and get

$$\begin{aligned}
\delta r_i &= -\left(\frac{n_0}{n_i}\right)^2 a_i\cos(2\lambda_i - 2\lambda_0)\\
\delta\theta_i &= \frac{11}{8}\left(\frac{n_0}{n_i}\right)^2\sin(2\lambda_i - 2\lambda_0). \qquad (8.1)
\end{aligned}$$

In the motion of the Galilean satellites, these inequalities are very small since $n_i \gg n_0$. The values of the coefficients are shown in Table 8.l.

Table 8.1. The Variations

Satellite	Coefficients of the Variations	
	Radius Vector	Longitude
I	–0.17 a_1 $\times 10^{-6}$	0.23 $\times 10^{-6}$
II	–0.67 a_2	0.93
III	–2.74 a_3	3.77
IV	–15.0 a_4	20.7

In the calculations, we adopted

$$\begin{aligned} n_0 &= 0.001\ 450\ 215\ \mathrm{d}^{-1} \\ a_0 &= 10900.84\ b \\ e_0 &= 0.04842 \\ m_0 &= 1047.572 \text{ Jupiter masses;} \end{aligned}$$

the usual value 1047.355 is the ratio of the mass of the Sun to that of the Jovian system, while m_0 is the ratio of the mass of Sun to that of Jupiter.

We may compare with the theory of the Moon where $n_0/n_i \simeq 1/13$ and the variation in longitude is close to 0.5 arc degree.

8.2 The Annual Equations

The *annual equations* are the inequalities whose argument is the mean anomaly of Jupiter in its motion around the Sun. The period of these inequalities is the anomalistic period of Jupiter; in the theory of the Moon, the period is one anomalistic year and this is the reason of the denomination annual.

The annual equations arise from

$$\frac{3}{4}\frac{Gm_0}{a_0}\left(\frac{a_i}{a_0}\right)^2 e_0 \cos(\lambda_0 - \varpi_0)$$

which forms part of the disturbing function and contributes only to one of the variational equations:

$$\frac{d\varepsilon_i^{\mathrm{I}}}{dt} = -3\frac{Gm_0}{n_i a_0^3} e_0 \cos(\lambda_0 - \varpi_0)$$

which on integration gives

$$\delta\theta_i = \delta\varepsilon_i^{\mathrm{I}} = -3\frac{Gm_0}{n_i n_0 a_0^3} e_0 \sin(\lambda_0 - \varpi_0).$$

If we use $Gm_0 = n_0^2 a_0^3$, we get

$$\delta\theta_i = -3\frac{n_0}{n_i} e_0 \sin(\lambda_0 - \varpi_0).$$

This inequality has a very long period (12 years) and thus will be strongly affected by libration as discussed in Section 7.6. The formulae 7.14 apply to this case also and lead to

$$\delta\theta_i = -3n_0 e_0 \left(\frac{1}{n_i} + Q_i L \frac{C_1}{n_0^2 - C_1}\right) \sin(\lambda_0 - \varpi_0), \tag{8.2}$$

where

$$L = \frac{1}{n_1} - \frac{3}{n_2} + \frac{2}{n_3}.$$

The numerical results depend on the values of C_1 and also on the factors of libration Q_i. If the Q_i are considered as in Section 7.1, and if we take $C_1 = 9.18 \times 10^{-6}$, which corresponds to $P = 2074$ d, we have the results shown in Table 8.2.

Table 8.2. Annual Equations

Satellite	Coefficients	
	libration corrected	without correction
I	–3.0 $\times 10^{-5}$	–5.9 $\times 10^{-5}$
II	–18.4	–11.9
III	–23.4	–24.0
IV	–56.0	–56.0

8.3 The Evections

The *evections* are the inequalities that arise from the term

$$\frac{15}{8}\frac{Gm_0}{a_0}\left(\frac{a_i}{a_0}\right)^2 e_i^2 \cos(2\lambda_0 - 2\varpi_i).$$

Because of the squared eccentricity that appears as a factor, the only significant contributions from this term come from perturbations in the eccentricities and perijoves. We have to add the term

$$-\frac{15Gm_0}{4n_i a_0^3} e_i \exp \mathrm{i}(2\lambda_0 - \varpi_i)$$

or

$$-\frac{15Gm_0}{4n_i a_0^3} \zeta_i^* \exp 2\mathrm{i}\lambda_0 \tag{8.3}$$

to the right-hand side of equations (5.5); in (8.3), the asterisk stands for complex conjugation. It is worth recalling that in Chapters V and VI we already found

$$\zeta_i = \sum_\mu M_i^\mu \exp \mathrm{i}(g^\mu t + \beta^\mu) + \overline{B}_i \exp \mathrm{i}u.$$

This approximation allows us to modify (8.3) to

$$-\frac{15Gm_0}{4n_i a_0^3}\left(\sum_\mu M_i^\mu \exp \mathrm{i}(2\lambda_0 - g^\mu t - \beta^\mu) + \overline{B}_i \exp \mathrm{i}(2\lambda_0 - u)\right).$$

The particular solutions that correspond to this additive term are

$$\zeta_j = -\sum_\mu A_j^\mu \exp \mathfrak{i}(2\lambda_0 - g^\mu t - \beta^\mu) - \overline{A}_j \exp \mathfrak{i}(2\lambda_0 - u)$$

where the real coefficients A_j^μ and $\overline{A}_j$ are to be determined. The substitution of this solution in the complete equation leads to five algebraic systems:

$$\begin{aligned}(2n_0 - g^\mu)A_j^\mu + \sum_k \{j,k\}A_k^\mu &= -\frac{15Gm_0}{4n_j a_0^3} M_j^\mu \qquad (\mu = 1,2,3,4)\\ (2n_0 + \mathfrak{m})\overline{A}_j + \sum_k \{j,k\}\overline{A}_k &= -\frac{15Gm_0}{4n_j a_0^3} \overline{B}_j.\end{aligned}$$

Since the evections have small amplitudes, we may consider the approximate solutions obtained when the terms outside the main diagonal in $\{j,k\}$ are neglected, i.e., $\{j,k\} = 0$ if $j \neq k$. This approximation yields

$$\begin{aligned}A_j^\mu &= -\frac{15Gm_0}{4n_j a_0^3} \frac{M_j^\mu}{2n_0 - g^\mu + \{j,j\}}\\ \overline{A}_j &= -\frac{15Gm_0}{4n_j a_0^3} \frac{\overline{B}_j}{2n_0 + \mathfrak{m} + \{j,j\}}.\end{aligned}$$

In some cases, however, the divisor $2n_0 - g^\mu + \{j,j\}$ approaches zero and a rigorous solution of the algebraic system is needed. If the result is enhanced by an internal resonance, the theory of the evections must be reformulated to get meaningful results.

The evections in longitude and radius vector are

$$\begin{aligned}\delta r_j &= \sum A_j^\mu a_j \cos(2\lambda_0 - \lambda_j - g^\mu t - \beta^\mu) + \overline{A}_j a_j \cos(2\lambda_0 - \lambda_j - u)\\ \delta\theta j &= \sum 2A_j^\mu \sin(2\lambda_0 - \lambda_j - g^\mu t - \beta^\mu) + 2\overline{A}_j \sin(2\lambda_0 - \lambda_j - u).\end{aligned}$$

At variance with the theory of the Moon, we have four evections for each satellite rather than just one. Each evection is related to one of the proper perijoves. Also, there is a fifth evection related to the induced equations of the centre. The numerical results are shown in Table 8.3.

Table 8.3. Coefficients of the Evections (units:10^{-6})

j	A_j^1/M_1^1	A_j^2/M_2^2	A_j^3/M_3^3	A_j^4/M_4^4	$\overline{A}_j$
1	+1038	–616	–82	–1	–0.7
2	+198	–2735	–267	–21	+2.7
3	+1711	+201	– 3395	–299	–0.3
4	–123	+1	+822	–7381	0

References and Notes

- 8.1
  Jupiter orbital elements are from
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- 8.2
  Second-order terms increase the evection of Jupiter IV (Callisto) by apr. 1 arcsecond.
  S.Ferraz-Mello: 1968, "Sur l'Evection de Callisto dans la Théorie de Laplace", *Anais Acad. Brasil. Ciências* 40, 447-449.

9

The Rotation of Jupiter

9.1 Euler's Dynamical Equations

The motion of a rigid body is completely known if we know the space motion of one fixed point inside the body and the motion of the body around this fixed point. The theory of motion of a rigid body depends on two main theorems: (i) the theorem of (linear) momentum ($\boldsymbol{Q}$) and (ii) the theorem of angular momentum ($\boldsymbol{s}$):

$$\frac{d\boldsymbol{Q}}{dt} = \boldsymbol{F}_{\text{ext}} \qquad\qquad \frac{d\boldsymbol{s}}{dt} = \boldsymbol{M}_{\text{ext}}$$

where $\boldsymbol{F}$ and $\boldsymbol{M}$ are the total external forces and the total moment (torque) of the external forces acting on the body with respect to the point of reference. If the motion is a free motion (i.e. without constraints), it is convenient to take the point of reference in the centre of mass of the body. In this Chapter, we assume that the space motion of the centre of mass is well known and we study the motion of the solid around the centre of mass.

Euler's dynamical equations are the most suitable expressions of the theorem of the angular momentum. The angular momentum is defined by

$$\boldsymbol{s} = \int_M (\boldsymbol{r} \times \boldsymbol{v}) dm$$

where the integral is taken over the complete mass of the body; the velocity $\boldsymbol{v}$ of a point is related to the velocity of the centre of mass through the expression

$$\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{\omega} \times \boldsymbol{r}$$

in which $\boldsymbol{\omega}$ is the instantaneous angular velocity vector. We have

$$\boldsymbol{s} = \int_M \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}) dm.$$

If the reference system is formed by the principal axes of inertia of the body, it follows

$$\boldsymbol{s} = Ap\,\boldsymbol{i} + Bq\boldsymbol{j} + Cr\boldsymbol{k}$$

where A, B, C are the moments of inertia along the principal axes, p, q, r are the components of $\boldsymbol{\omega}$ in this system of axes and $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are three unit vectors that form a right-handed system along the principal axes. The derivative of $\boldsymbol{s}$ is

$$\frac{d\boldsymbol{s}}{dt} = \dot{\boldsymbol{s}} + \boldsymbol{\omega} \times \boldsymbol{s}.$$

Euler's dynamical equations in vector form are

$$\dot{\boldsymbol{s}} + \boldsymbol{\omega} \times \boldsymbol{s} = \boldsymbol{M}_{\text{ext}};$$

in the classical scalar form, they are

$$\begin{aligned} A\dot{p} - (B - C)qr &= L \\ B\dot{q} - (C - A)rp &= M \\ C\dot{r} - (A - B)pq &= N \end{aligned} \tag{9.1}$$

where L, M, N are the projections of lhe moment of the external forces with respect to the centre of mass, on the principal axes.

9.2 Free Nutation of Jupiter

Let the rigid body under consideration be Jupiter and let the external actions be Newtonian actions arising from another body, for example, one of its satellites. Let the distribution of mass of Jupiter be assumed axially symmetric, that is, $B = A$. The force that acts on an element of mass dm in the planet is

$$d\boldsymbol{f}_i = -\frac{Gm_i dm}{|\boldsymbol{r} - \boldsymbol{r}_i|^3}(\boldsymbol{r} - \boldsymbol{r}_i)$$

where m_i is the external mass, $\boldsymbol{r}_i$ is its jovicentric position and $\boldsymbol{r}$ is the jovicentric position of the element of mass. The moment of this force about the centre of mass of the planet is

$$d\boldsymbol{M}_i = \boldsymbol{r} \times d\boldsymbol{f}_i$$

and hence the total moment

$$\boldsymbol{M} = \int_M \boldsymbol{r} \times d\boldsymbol{f}_i$$

or

$$\boldsymbol{M} = -\boldsymbol{r}_i \times \text{grad}_{\mathsf{P}_i} W_i \tag{9.2}$$

where

$$W_i = -\int_M \frac{Gm_i dm}{|\boldsymbol{r} - \boldsymbol{r}_i|}$$

is the force-function of the gravitational field of Jupiter at the point P_i and it was considered in Section 3.1 (equation 3.2). Except for the central part Gm_i/r_i whose contribution to the cross product in equation (9.2) is zero, and except for the factor m_i that is not considered since R_{iJ} is a force-function per unit mass (i.e. an acceleration-function), W_i and R_{iJ} are the same. Then

$$W_i = -\frac{Gm_i J_2}{r_i^3} P_2(\sin\phi_i). \tag{9.3}$$

It is noteworthy that $b = 1$ and $M_J = 1$ are natural units in this text. The z-component of the moment is given by

$$N = \frac{\partial W_i}{\partial X_i} Y_i - \frac{\partial W_i}{\partial Y_i} X_i$$

where X_i, Y_i, Z_i are the coordinates of the external body referred to the axes of inertia of Jupiter. On account of the fact that $\sin\phi_i = Z_i/r_i$, W_i depends on X_i and Y_i only through r_i. Thus,

$$N = \frac{\partial W_i}{\partial r_i} \left(\frac{X_i}{r_i} Y_i - \frac{Y_i}{r_i} X_i \right) = 0$$

and the third Euler's dynamical equation becomes

$$\dot{r} = 0.$$

Thus, if the planet is rigid and has axially symmetric mass distribution, we have r=constant. In other words, the polar component of the rotation vector is constant; the remaining equations become

$$\begin{aligned} \dot{p} + \nu q &= \frac{L}{A} \\ \dot{q} - \nu p &= \frac{M}{A} \end{aligned} \tag{9.4}$$

where

$$\nu = \frac{C - A}{A} r.$$

If the body is free from external forces (that is, if $L = M = 0$), the immediate solutions are

$$\begin{aligned} p &= \alpha r \cos(\nu t + \beta) \\ q &= \alpha r \sin(\nu t + \beta) \end{aligned} \tag{9.5}$$

where α and β are integration constants. The rotation vector in this free motion describes a circular cone about the symmetry axis whose opening depends on the initial conditions and whose period is

$$T = \frac{2\pi}{r}\frac{A}{C-A}. \tag{9.6}$$

The dynamical determination of J_2 allows us to calculate $C-A = 0.01475$; since $A < 0.4$ (limit that corresponds to a homogeneous distribution), there results $T < 11$ days. The amplitude shall be very small. It is worth remembering that for Earth, the observed Chandler period is around 14 months (for a rigid Earth it would be 305 days), and the amplitude corresponds to $\alpha = 2 \times 10^{-6}$.

9.3 Euler's Geometric Equations

The knowledge of the forced motion of the planet requires particular solutions of the complete equations (9.4).

Let a set of geometrical equations relating a reference system solidary with the body and the fundamental system of reference be considered. Let $\boldsymbol{K}$ be a unit vector in the direction of the pole of the fundamental reference system. In a moving reference system, the time derivative of $\boldsymbol{K}$ is

$$\dot{\boldsymbol{K}} + \boldsymbol{\omega} \times \boldsymbol{K} = 0. \tag{9.7}$$

Let the components of $\boldsymbol{K}$ in the moving reference system be K_1, K_2, K_3. The scalar equivalents of equation (9.7) are Poisson's equations:

$$\begin{aligned} \dot{K}_1 &= rK_2 - qK_3 \\ \dot{K}_2 &= pK_3 - rK_1 \\ \dot{K}_3 &= qK_1 - pK_2 \end{aligned} \tag{9.8}$$

Also (see figure), $\boldsymbol{K} \cdot \boldsymbol{k} = \cos \widetilde{I}$ and

$$\mathrm{proj}_{ij}\boldsymbol{K} = \sin \widetilde{I}(-\sin\chi \boldsymbol{i} + \cos\chi \boldsymbol{j})$$

where the angle χ is measured from a fixed meridian on the equator and, by definition, includes the rotation of the planet. The last equations become

$$\begin{aligned} K_1 &= -\sin \widetilde{I} \sin\chi \\ K_2 &= \sin \widetilde{I} \cos\chi \\ K_3 &= \cos \widetilde{I}. \end{aligned} \tag{9.9}$$

It is evident that the determinant of the system (9.8) where the unknowns are r, p and q, is zero and the system cannot be solved. Under such condition an independent additional equation is necessary. On equating the derivatives of the unit-vector $\boldsymbol{N}$ in both the system – moving and fixed – we have

$$\dot{\boldsymbol{N}} + \boldsymbol{\omega} \times \boldsymbol{N} = \dot{\boldsymbol{\Omega}}(\boldsymbol{K} \times \boldsymbol{N}) \tag{9.10}$$

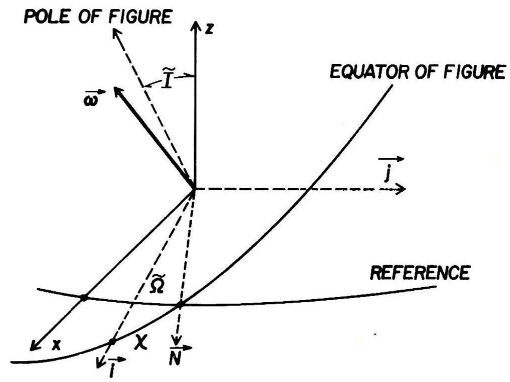

Fig. 9.1.

and, from figure 9.1,

$$\begin{aligned} N_1 &= \cos\chi \\ N_2 &= \sin\chi \\ N_3 &= 0. \end{aligned}$$

The first scalar component of equation (9.10) is

$$\dot{N}_1 = rN_2 - \dot{\tilde{\Omega}} K_3 N_2.$$

Since $\dot{N}_1 = -\dot{\chi} N_2$, we have

$$r = K_3 \dot{\tilde{\Omega}} - \dot{\chi}. \tag{9.11}$$

Equations (9.8) may now be solved for p and q. Keeping in mind that

$$\begin{aligned} \dot{K}_1 &= -K_2\dot{\chi} - K_3 \dot{\tilde{I}} \sin\chi \\ \dot{K}_2 &= K_1\dot{\chi} - K_3 \dot{\tilde{I}} \cos\chi, \end{aligned}$$

we get

$$\begin{aligned} p &= K_1 \dot{\tilde{\Omega}} + \cos\chi\ \dot{\tilde{I}} \\ q &= K_2 \dot{\tilde{\Omega}} + \sin\chi\ \dot{\tilde{I}}. \end{aligned} \tag{9.12}$$

Equations (9.11) and (9.12) are Euler's geometric equations.

9.4 Equations of Motion of Jupiter's Equator

On substitution of Euler's geometric equations in Euler's dynamical equations (9.4) and considering equations (9.8) and (9.11), we have

$$A\big(-qK_3\dot{\widetilde{\Omega}}+K_1\ddot{\widetilde{\Omega}}-K_3\dot{\widetilde{\Omega}}\dot{\widetilde{I}}\sin\chi+\ddot{\widetilde{I}}\cos\chi\big)+Cr\big(K_2\dot{\widetilde{\Omega}}+\dot{\widetilde{I}}\sin\chi\big)=L$$

$$A\big(\;pK_3\dot{\widetilde{\Omega}}+K_2\ddot{\widetilde{\Omega}}+K_3\dot{\widetilde{\Omega}}\dot{\widetilde{I}}\cos\chi+\ddot{\widetilde{I}}\sin\chi\big)-Cr\big(K_1\dot{\widetilde{\Omega}}+\dot{\widetilde{I}}\cos\chi\big)=M.$$

Since all velocities in the left-hand sides are negligible when compared to r, we can write, approximately,

$$Cr\big(K_2\dot{\widetilde{\Omega}}+\dot{\widetilde{I}}\sin\chi\big)=L$$

$$-Cr\big(K_1\dot{\widetilde{\Omega}}+\dot{\widetilde{I}}\cos\chi\big)=M.$$

Solving these equations for $\dot{\widetilde{\Omega}}$ and $\dot{\widetilde{I}}$ and substituting equations (9.9) in the results, we get

$$\begin{aligned}\dot{\widetilde{\Omega}}\sin\widetilde{I} &= \frac{L}{Cr}\cos\chi+\frac{M}{Cr}\sin\chi\\ \dot{\widetilde{I}} &= \frac{L}{Cr}\sin\chi-\frac{M}{Cr}\cos\chi.\end{aligned}\tag{9.13}$$

The components L and M of the moment of the external forces defined by equations (9.2) and (9.3), may be written as

$$L=-YW_Z$$

$$M=\quad XW_Z$$

where

$$W_Z=-\frac{3Gm_iJ_2}{r_i^5}Z_i=-\frac{3Gm_iJ_2}{r_i^4}\sin\phi_i$$

is the partial derivative of W with respect to Z_i, obtained explicitly and not considering the dependence of W on Z_i through r_i, for it does not contribute to L and M. Substitution in equations (9.13) yields

$$\begin{aligned}\dot{\widetilde{\Omega}}\sin\widetilde{I} &= -\frac{1}{Cr}W_Z(-X\sin\chi+Y\cos\chi)\\ \dot{\widetilde{I}} &= -\frac{1}{Cr}W_Z(X\cos\chi+Y\sin\chi).\end{aligned}\tag{9.14}$$

The brackets represent a rotation of the solidary axes that brings the x-axis into the intersection (nodal line) of the fundamental reference plane and the equator of the planet. The rotation of the system about this line bringing the fundamental reference plane to coincide with the equator results in a correction of the order of $(1-\cos\widetilde{I})$, that is, of the order of the square of the inclinations. Neglecting this correction, we have

$$\begin{aligned}\dot{\widetilde{\Omega}}\sin\widetilde{I} &= -\frac{1}{Cr}W_Z\overline{y}_i\\ \dot{\widetilde{I}} &= -\frac{1}{Cr}W_Z\overline{x}_i.\end{aligned}\tag{9.15}$$

The values of $\overline{x}_i$ and $\overline{y}_i$ are given in Chapter III (equations 3.4) except for the correction of the fact that the origin of the system $\overline{x}, \overline{y}$ is displaced from the origin of the system x, y and this displacement is measured by the angle $\widetilde{\Omega}$. Thus

$$\overline{x}_i = r_i \left(\overline{x}(\theta_i - \widetilde{\Omega}) + \frac{1}{2} I_i^2 \sin(\Omega_i - \widetilde{\Omega}) \sin(\lambda_i - \Omega_i) \right)$$

$$\overline{y}_i = r_i \left(\sin(\theta_i - \widetilde{\Omega}) - \frac{1}{2} I_i^2 \cos(\Omega_i - \widetilde{\Omega}) \sin(\lambda_i - \Omega_i) \right).$$

Retaining only the terms that are independent of orbital eccentricity and inclination, we have the approximate re1ations

$$\overline{x}_i = a_i \cos(\lambda_i - \widetilde{\Omega})$$

$$\overline{y}_i = a_i \sin(\lambda_i - \widetilde{\Omega}).$$

The direction cosines of the axis OZ in the system $Oxyz$ can easily be calculated, and

$$Z_i = -\overline{y}_i \sin \widetilde{I} + z_i \cos \widetilde{I}.$$

Neglecting the higher-order correction $(1 - \cos \widetilde{I})$, we have

$$Z_i = z_i - \overline{y}_i \widetilde{I}$$

or

$$Z_i = a_i I_i \sin(\lambda_i - \Omega_i) - a_i \widetilde{I} \sin(\lambda_i - \widetilde{\Omega}).$$

When all these results are substituted in equations (9.15) and when terms involving eccentricities as a factor are neglected, those equations become

$$\begin{aligned}
\dot{\widetilde{\Omega}} \sin \widetilde{I} &= -\frac{3Gm_i J_2}{2Cra_i^3} \left(-I_i \cos(\Omega_i - \widetilde{\Omega}) + \widetilde{I} + I_i \cos(2\lambda_i - \Omega_i - \widetilde{\Omega}) - \widetilde{I} \cos(2\lambda_i - 2\widetilde{\Omega}) \right) \\
\dot{\widetilde{I}} &= -\frac{3Gm_i J_2}{2Cra_i^3} \left(I_i \sin(\Omega_i - \widetilde{\Omega}) - I_i \sin(2\lambda_i - \Omega_i - \widetilde{\Omega}) + \widetilde{I} \sin(2\lambda_i - 2\widetilde{\Omega}) \right).
\end{aligned}$$

All terms in the brackets have coefficients of the same order (the order of inclinations). The integration destroys the equality in orders since every term will be multiplied by its period. To an approximation, it is sufficient to consider only the constant and long-period terms:

$$\begin{aligned}
\dot{\widetilde{\Omega}} \sin \widetilde{I} &= -\frac{3Gm_i J_2}{2Cra_i^3} \left(\widetilde{I} - I_i \cos(\Omega_i - \widetilde{\Omega}) \right) \\
\dot{\widetilde{I}} &= -\frac{3Gm_i J_2}{2Cra_i^3} I_i \sin(\Omega_i - \widetilde{\Omega}).
\end{aligned}$$

These equations may still be written as

$$\dot{\widetilde{\Omega}} = \frac{1}{Cr\sin\widetilde{I}}\frac{\partial U_i}{\partial\widetilde{I}}$$

$$\dot{\widetilde{I}} = \frac{-1}{Cr\sin\widetilde{I}}\frac{\partial U_i}{\partial\widetilde{\Omega}}$$

where

$$U_i = -\frac{3Gm_iJ_2}{4a_i^3}\left(\widetilde{I}^2 - 2I_i\widetilde{I}\cos(\Omega_i - \widetilde{\Omega})\right).$$

For Jupiter, instead of considering U_i, we have to consider $U = \sum U_i$ where the summation comprehends all the Galilean satellites and the Sun. Thus, these equations are considered in the form

$$\begin{aligned} \frac{d\widetilde{I}}{dt} &= -\sum_{j=0}^{4} W_j I_j \sin(\Omega_j - \widetilde{\Omega}) \\ \widetilde{I}\,\frac{d\widetilde{\Omega}}{dt} &= W_5\widetilde{I} + \sum_{j=0}^{4} W_j I_j \cos(\Omega_j - \widetilde{\Omega}) \end{aligned} \qquad (9.16)$$

where

$$W_j = \frac{3Gm_jJ_2}{2Cra_j^3}$$

and

$$W_5 = -\sum_{j=0}^{4} W_j.$$

10

Inequalities in Latitude

10.1 Variational Equations

The approximate variational equations for the spatial orbital elements are

$$\frac{dI_j}{dt} = -\frac{1}{n_j a_j^2 I_j}\frac{\partial R}{\partial \Omega_j} \qquad \frac{d\Omega_j}{dt} = \frac{1}{n_j a_j^2 I_j}\frac{\partial R}{\partial I_j}.$$

Since the inclinations of the orbits of the Galilean satellites are very small the above pair of equations will be considered in its modified form

$$\frac{dp_j}{dt} = \frac{1}{n_j a_j^2}\frac{\partial R}{\partial q_j} \qquad \frac{dq_j}{dt} = -\frac{1}{n_j a_j^2}\frac{\partial R}{\partial p_j}. \tag{10.1}$$

The problem is greatly simplified if we introduce a complex quantity Π_j defined by

$$\Pi_j = q_j + \mathrm{i} p_j \qquad \mathrm{i} = \sqrt{-1}.$$

Equations (10.1) become

$$\frac{d\Pi_j}{dt} = -\frac{1}{n_j a_j^2}\left(\frac{\partial R}{\partial p_j} - \mathrm{i}\frac{\partial R}{\partial q_j}\right) = \frac{2\mathrm{i}}{n_j a_j^2}\frac{\partial R}{\partial \Pi_j^*} \tag{10.2}$$

or

$$\frac{d\Pi_j}{dt} = \frac{\mathrm{i}}{n_j a_j^2}\left(\frac{\partial R}{\partial I_j} + \frac{\mathrm{i}}{I_j}\frac{\partial R}{\partial \Omega_j}\right)\exp \mathrm{i}\Omega_j. \tag{10.3}$$

Let us consider the spatial parts of the disturbing functions where the longitudes of the satellites and of the Sun are absent. We have

$$-\frac{3}{8}\frac{Gm_0}{a_0}\left(\frac{a_j}{a_0}\right)^2 \left(I_0^2 + I_j^2 - 2I_0 I_j \cos(\Omega_i - \Omega_0)\right)$$
$$-\frac{3}{4}\frac{GJ_2}{a_j^3}\left(\widetilde{I}^2 + I_j^2 - 2\widetilde{I} I_j \cos(\Omega_j - \widetilde{\Omega})\right)$$
$$-\frac{1}{8}\sum_{k\neq j} Gm_k B_{jk}^1 \left(I_j^2 + I_k^2 - 2I_j I_k \cos(\Omega_j - \Omega_k)\right).$$

The corresponding variational equations are

$$\frac{d\Pi_j}{dt} = -\frac{G}{4n_j a_j^2}\left(\frac{3m_0 a_j^2}{a_0^3} + \frac{6J_2}{a_j^3} + \sum_k m_k B_{jk}^1\right)\mathrm{i}\Pi_j \qquad (10.4)$$
$$+\frac{3Gm_0}{4n_j a_0^3}\mathrm{i}\Pi_0 + \frac{3GJ_2}{2n_j a_j^5}\mathrm{i}\Pi_5 + \sum_k \frac{Gm_k B_{jk}^1}{4n_j a_j^2}\mathrm{i}\Pi_k$$

where, to keep similarity, we put

$$\Pi_0 = I_0 \exp \mathrm{i}\Omega_0 \qquad\qquad \Pi_5 = \widetilde{I} \exp \mathrm{i}\widetilde{\Omega}. \qquad (10.5)$$

If equations (10.4) are compared to equations (9.16), we find that all these equations are interdependent and may not be integrated separately. Thus, besides equations (10.4), we have to consider

$$\frac{d\Pi_5}{dt} = \sum_0^5 \mathrm{i}W_k \Pi_k. \qquad (10.6)$$

Equations (10.4) and (10.6) form a linear differential system

$$\frac{d\Pi_j}{dt} - \mathrm{i}\sum_{k=1}^{5}(j,k)\Pi_k = \mathrm{i}(j)\Pi_0 \qquad (10.7)$$

where, for each $j, k = 1, 2, 3, 4$,

$$\begin{aligned}
(j,j) &= -\frac{G}{4n_j a_j^2}\left(\frac{3m_0 a_j^2}{a_0^3} + \frac{6J_2}{a_j^3} + \sum_k m_k B_{jk}^1\right) \\
(5,5) &= W_5 \\
(j,k) &= \frac{Gm_k B_{jk}^1}{4n_j a_j^2} \qquad\qquad (j \neq k) \\
(j,5) &= \frac{3GJ_2}{2n_j a_j^5} \\
(5,k) &= W_k = \frac{3Gm_k J_2}{2Cr a_k^3} \\
(j) &= \frac{3Gm_0}{4n_j a_0^3} \\
(5) &= W_0 = \frac{3Gm_0 J_2}{2Cr a_0^3}.
\end{aligned}$$

The numerical values of (j,k) and (j) are given in Table 10.1. For the main-diagonal elements and for the elements in the fifth column, we considered also the second-order contributions

$$\frac{15G}{8n_j a_j^7}(3J_2^2 - 2J_4)$$

that were added to the $(5, j)$ and subtracted from the (j, j).

It is noteworthy that in each row of Table 10.1, the sum is equal to zero. Indeed, from the preceding formulae, we have

$$\sum_{k=1}^{5}(j,k) + (j) = 0. \tag{10.8}$$

This is a well-known fact in planetary theories and it is not altered when the motion of the equatorial plane of the primary body is included. The only difference with respect to the classical equations of p1anetary theory is the definition of the (j, k) when j or k takes value 5 and the existence of an external action, which gives rise to non-zero coefficients (j).

Table 10.1. Values of (j, k) and (j) (in units $10^{-7}\mathrm{d}^{-1}$)

j	$(j,1)$	$(j,2)$	$(j,3)$	$(j,4)$	$(j,5)$	(j)
1	-23262	438	175	18	22626	4
2	638	-5789	659	43	4440	9
3	66	170	-1261	142	865	18
4	7	12	150	-330	120	42
5	33	4	3	0	-42	0

Another important property of (j, k) is that

$$d_k\,(k,j) = d_j\,(j,k) \tag{10.9}$$

where

$$d_j = m_j n_j a_j^2 \qquad (j < 5), \qquad\qquad d_5 = Cr.$$

Let the quantity

$$s = \sum_j d_j \Pi_j$$

be defined. From equation (10.7), it follows

$$\dot{s} = \mathrm{i}\sum_j d_j \sum_k (j,k)\Pi_k + \mathrm{i}\sum_j d_j (j)\Pi_0$$

and, using the symmetry property (10.9), we have

$$\dot{s} = \mathrm{i}\sum_k d_k \Pi_k \sum_j (k,j) + \mathrm{i}\sum_j d_j (j)\Pi_0.$$

Hence

$$\dot{s} = \mathfrak{i} \sum_j d_j(j)(\Pi_0 - \Pi_j).$$

If the solar influence is not considered, the right-hand side of the above equation vanishes and we find the law of conservation of the angular momentum: $s = \text{const}$. The existence of this first integral is associated with the fact that in absence of external (solar) actions, we have $\det(j,k) = 0$ and one of the characteristic roots of the system is zero.

In planetary theory, the plane whose inclination and node are given by $\Pi = s / \sum d_j$ is called the *invariable plane*. This plane is no more invariable in the Galilean system since the (j) cannot be taken as zero.

10.2 Free Oscillation of the Nodes

In these calculations, like in the calculation of the free equations of the centre (Section 5.2), we cannot accept approximate free solutions obtained from the separated equations

$$\frac{d\Pi_j}{dt} - \mathfrak{i}(j,j)\Pi_j = 0. \tag{10.10}$$

The solution is obtained by integrating the complete associated homogeneous system

$$\frac{d\Pi_j}{dt} - \mathfrak{i} \sum_{k=1}^{5} (j,k)\Pi_k = 0 \tag{10.11}$$

whose fundamental solutions are the functions

$$\Pi_j = C_j \exp \mathfrak{i} bt$$

where b is a root of the characteristic polynomial

$$\det\big(b\,\delta_{jk} - (j,k)\big) = 0.$$

The comments of Section 5 2 also apply in this case and the general solution of equation (10.11) is

$$\Pi_j = \sum N_j^\mu \exp \mathfrak{i}(b^\mu t + \gamma^\mu) \tag{10.12}$$

The values of the four main characteristic roots obtained by the values listed in Table 10.1 are given in Table 10.2 where we compare these values with those obtained by Sampson, de Sitter, Brown and Lieske.

The angles $b^\mu t + \gamma^\mu$ $(\mu < 5)$ are the longitudes of the *proper nodes*. The assignment of a proper node to a satellite is made without ambiguity since the solutions in the non-coupled case would be $b^\mu = (\mu, \mu)$ and the numerical difference to the actual values is small.

Table 10.2. The Characteristic Roots (in units $10^{-6}\mathrm{d}^{-1}$)

μ	b^μ	Sampson	de Sitter	Brown	Lieske
1	−2331	−2340	−2306±16	−2305	−2318
2	−580	−571	−569± 7	−576	−569
3	−126	−123	−124± 1	−123	−125
4	−31	−32	−31± 1	−31	−31

The fifth root is extremely small:

$$b^5 = -3.3 \times 10^{-8}\mathrm{d}^{-1}$$

(Souillart found $-3.8 \times 10^{-8}\mathrm{d}^{-1}$).

The real constants N_j^μ are not independent, and are such that:

$$\sum_{k=1}^{5} (b^\mu \, \delta_{jk} - (j,k)) \, N_k^\mu = 0. \tag{10.13}$$

To each value of μ, we have five equations out of which four are independent. The N_j^μ are completely known if we know 5 amongst them, one for each value of μ. With the numerical values listed in Table 10.1, we obtain, in units of the corresponding N_μ^μ, the set of values shown in Table 10.3.

Table 10.3. The Eigenvectors (units N_μ^μ)

μ	N_1^μ	N_2^μ	N_3^μ	N_4^μ	N_5^μ
1	1	−0.0359	−0.0027	−0.0003	−0.0014
2	0.0236	1	−0.0377	−0.0011	−0.0009
3	0.0071	0.1418	1	−0.1620	−0.0034
4	−0.0013	0.0227	0.1493	1	−0.0038
5	0.9994	0.9939	0.9695	0.8601	1

These values may be compared with the values recently obtained by Brown which are thrown in Table 10.4.

Table 10.4. Brown's Eigenvectors (units N_μ^μ)

μ	N_1^μ	N_2^μ	N_3^μ	N_4^μ
1	1	−0.0351	−0.0026	-
2	0.0244	1	−0.0370	−0.0011
3	0.0111	0.1499	1	−0.1745
4	0.0022	0.0241	0.1367	1

By analogy with the approximated solutions obtained from the separated equations (10.10), the N_j^j may be called *proper inclinations*. However, the way in which forced oscillations are considered in Section 10.3 modifies equations (10.12) and also the geometric interpretation of the integration constants (see Section 10.4).

10.3 Forced Oscillations

On the right-hand sides of equations (10.4), we have the terms $\mathfrak{i}(j)\Pi_0$ where Π_0 is the complex parameter that gives the position of the orbital plane of Jupiter. If we could assume it as a constant, the particular solutions of equations (10.8) would be $\Pi_j = \Pi_0$(constant) for all j, and the general solutions would be

$$\Pi_j = \Pi_0 + \sum_\mu N_j^\mu \exp \mathfrak{i}(b^\mu t + \gamma^\mu). \tag{10.14}$$

In fact, Π_0 would be a constant if Jupiter was the only planet in the Solar System. However, other planets disturb the orbital motion of Jupiter. With respect to the ecliptic and equinox of a certain date, say 1950.0, we have

$$\Pi_0(t) = \sum_\nu S^\nu \exp \mathfrak{i}(s^\nu t + \sigma^\nu)$$

and this function is nothing but the general solution of the system (11.11) in the problem of mutual interactions of the planets in the Solar System. The numerical values found by Brouwer and van Woerkom are given in Table 10.5

Table 10.5. Motion of Jupiter's Orbital Plane

ν	S^ν	s^ν	σ^ν (1950.0)
0	+ 0.0275703	0 $\times 10^{-8}\mathrm{d}^{-1}$	+1.869
1	− 207	−6.9	+0.339
2	− 130	+8.7	−0.732
3	− 2	−24.9	−1.832
4	− 18	−23.4	−1.108
5	− 63064	−34.2	+2.223
6	− 9571	−3.9	−0.784
7	− 11689	−0.9	−2.753

We have not yet defined the fixed reference plane. Let it be the orbital plane of Jupiter in a given epoch t_0. Then, from the spherical triangle formed by this plane, the orbital plane at t and the reference plane of Table 10.5 (see figure 10.1) neglecting terms that give a value of the order of 10^{-5}, we have

$$\Pi_0 = \Pi_0(t) - \Pi_0(t_0)$$

or

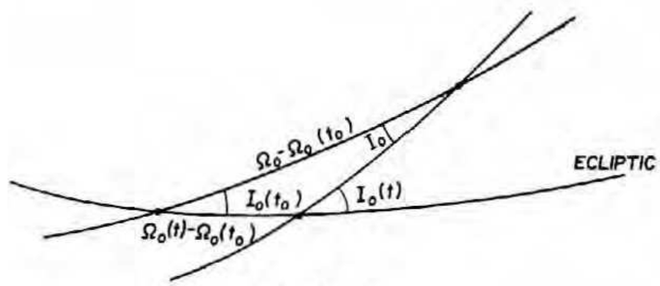

Fig. 10.1.

$$\Pi_0 = \sum_{\nu=0}^{7} S^\nu \big(\exp \mathrm{i}(s^\nu t + \sigma^\nu) - \exp \mathrm{i}(s^\nu t_0 + \sigma^\nu) \big).$$

For sake of convenience, we will take t_0 as the epoch of the data in Table 10.5, that is, 1950.0; therefore

$$\Pi_0 = \sum_{\nu=1}^{7} S^\nu \big(\exp \mathrm{i}(s^\nu t + \sigma^\nu) - \exp \mathrm{i}\sigma^\nu \big).$$

It is sufficient to consider the first-order approximation

$$\Pi_0 = \mathrm{i}t \sum_{\nu=1}^{7} S^\nu s^\nu \exp \mathrm{i}\sigma^\nu = \mathrm{i}K_0 t \tag{10.15}$$

as the period of all components are very great (the shortest is that for $\nu = 5$ and is 50,000 years). The forced oscillations, considered in only a small fraction of their periods, will appear as secular and have the form

$$\delta\Pi_j = \Pi_j^0 + \mathrm{i}\Lambda_j t \tag{10.16}$$

where Π_j^0 and Λ_j are undetermined coefficients. If this solution is introduced in the complete equations (10.7), the identification in the powers of t yields

$$\Lambda_j - \sum (j,k)\Pi_k^0 = 0$$

$$\sum (j,k)\Lambda_k = -(j)K_0.$$

By using the equation (10.8), it follows

$$\Lambda_k = K_0$$

for all k, and

$$K_0 - \sum_k (j,k)\Pi_k^0 = 0. \tag{10.17}$$

The numerical results are

$$\begin{aligned} \Pi_1^0 &= 0.99935\Pi_5^0 - 6K_0 \times 10^2 \\ \Pi_2^0 &= 0.9938\Pi_5^0 - 35K_0 \times 10^2 \\ \Pi_3^0 &= 0.9692\Pi_5^0 - 125K_0 \times 10^2 \\ \Pi_4^0 &= 0.859\Pi_5^0 - 360K_0 \times 10^2 \end{aligned} \tag{10.18}$$

The complete system may not be solved because the fifth equation introduces a huge uncertainty in the system. Roughly we have $\Pi_5^0 = -3K_0 \times 10^7$. The error in Π_5^0 does not allow to give different values for the other Π_j^0; these errors are large and arise mainly from the fact that (5) is very close to zero. A better determination of Π_5^0 needs the accurate determination of satellite masses with at least 4 significant figures.

We may relate Π_5^0 to N_k^μ. Equations (10.17) and (10.13) when multiplied by $d_j N_j^\mu$ and $d_j \Pi_j^0$, respectively, and summed over the subscript j, yield

$$\sum_j \sum_k d_j N_j^\mu (j,k) \Pi_k^0 = K_0 \sum_j d_j N_j^\mu$$

$$\sum_j \sum_k d_j \Pi_j^0 (j,k) N_k^\mu = b^\mu \sum_j d_j N_j^\mu \Pi_j^0 .$$

Because of equation (10.9), the left-hand sides of the above equations are equal, then

$$b^\mu \sum_j d_j N_j^\mu \Pi_j^0 = K_0 \sum_j d_j N_j^\mu .$$

This system has 5 linear algebraic equations and may be solved. We have

$$\Pi_j^0 = \sum_\mu \frac{N_j^\mu}{b^\mu} \frac{\sum_k d_k N_k^\mu}{\sum_k d_k (N_k^\mu)^2} K_0 . \tag{10.19}$$

In view of the values listed in Table 10.3 together with the values of the characteristic roots, it is evident that only the term $\mu = 5$ contributes significantly to Π_5^0. With an internal precision better than 10^{-3}, we may write

$$\Pi_5^0 = \frac{K_0}{b^5},$$

which shows that the large uncertainty in the determination of the Π_j^0 is a consequence of the rough determination of the fifth characteristic root.

The particular solution of the complete system will be considered in a modified form. Indeed, we can add $\alpha N_j^5 \exp \mathrm{i} b^5 t$ with arbitrary constant α to any solution and the result is still a solution. Since b^5 is very small, even when compared to the other characteristic roots, this additive term may be written $\alpha N_j^5 (1 + \mathrm{i} b^5 t)$. Adding this term to the particular solutions (10.16) and choosing $\alpha N_j^5 = -\Pi_j^0$ in order to eliminate the constant term of $\delta\Pi_j$, we get the particular solutions

$$\delta\Pi_j = \mathrm{i}K_j t$$

where

$$K_j = \Lambda_j - \Pi_j^0 b^5.$$

The numerical values of the K_j are given by

$$\begin{aligned}
K_1 &= 0.00065K_0 = 0.014 \times 10^{-10} \exp \mathrm{i}\sigma \\
K_2 &= 0.0062K_0 = 0.13 \times 10^{-10} \exp \mathrm{i}\sigma \\
K_3 &= 0.0309K_0 = 0.65 \times 10^{-10} \exp \mathrm{i}\sigma \\
K_4 &= 0.141K_0 = 3.0 \times 10^{-10} \exp \mathrm{i}\sigma \\
K_5 &= 0
\end{aligned}$$

where we have introduced

$$K_0 = S_0 \exp \mathrm{i}\sigma$$

and used $S_0 = 2.12 \times 10^{-9}\mathrm{d}^{-l}$ and $\sigma = 2.23$ as calculated from Table 10.5.

The above choice of α is not the same as made by Souillart ($\alpha = 0$) or by Tisserand. Hence, the results of these calculations may not be directly compared.

10.4 Inequalities in Latitude. Proper Inclinations

Let ϕ_j' and ψ_j be the latitudes of a satellite referred to the fixed reference plane and to the orbital plane of Jupiter, respectively (see Figure 10.2). The sinus law yields

$$\sin\phi_j' = \sin I_j \sin(\theta_j - \Omega_j).$$

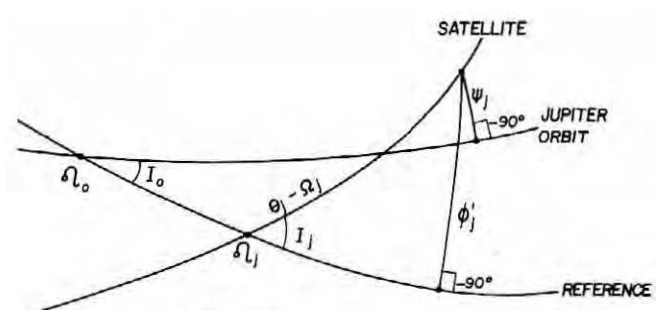

Fig. 10.2.

If higher-order terms and the satellite eccentricities are neglected, we have

$$\phi'_j = I_j \sin(\lambda_j - \Omega_j). \tag{10.20}$$

A similar reasoning over the other triangle yields

$$\phi'_j - \psi_j = I_0 \sin(\lambda_j - \Omega_0). \tag{10.21}$$

If we use the complex variables Π_j, the equations (10.20) and (10.21) become

$$\phi'_j = \Im(\Pi_j^* \exp \mathrm{i}\lambda_j),$$

$$\phi'_j - \psi_j = \Im(\Pi_0^* \exp \mathrm{i}\lambda_j).$$

Therefore

$$\psi_j = \phi'_j + tS_0 \cos(\lambda_j - \sigma).$$

Considering the results of the preceding sections, we have

$$\phi'_j = \sum_\mu N_j^\mu \sin(\lambda_j - b^\mu t - \gamma^\mu) - tS_j \cos(\lambda_j - \sigma).$$

$$\psi_j = \sum_\mu N_j^\mu \sin(\lambda_j - b^\mu t - \gamma^\mu) - t(S_0 - S_j) \cos(\lambda_j - \sigma)$$

where

$$S_j = \frac{K_j S_0}{K_0}.$$

To have the latitude referred to the equator of Jupiter, we use the approximate form of equation (3.6)

$$\phi_j = \phi'_j - \widetilde{I} \sin(\lambda_j - \widetilde{\Omega})$$

or

$$\phi_j = \Im\big((\Pi_j^* - \Pi_5^*) \exp \mathrm{i}\lambda_j\big).$$

Therefore

$$\phi_j = \sum_\mu (N_j^\mu - N_5^\mu) \sin(\lambda_j - b^\mu t - \gamma^\mu) - tS_j \cos(\lambda_j - \sigma). \tag{10.22}$$

$N_j^j - N_5^j$ are the *proper inclinations* of the orbital planes referred to the plane of Jupiter's equator. The smallness of the N_5^j ($j \neq 5$) and the inaccuracy that involves the determination of the N_j^j allow us to write $N_j^j - N_5^j \simeq N_j^j$.

The proper inclinations determined by Sampson, de Sitter and Lieske are gathered in Table 10.6.

Table 10.6. Proper Inclinations (in units 10^{-4})

Satellite	Sampson	de Sitter	Lieske
1	4.76	5.5±0.2	7.0±2.9
2	81.55	81.4±0.2	81.5±2.0
3	35.59	31.2±0.2	32.4±1.6
4	47.47	42.8±0.2	44.3±4.8

10.5 Position of Jupiter's Equator

The position of the equator of Jupiter with respect to the reference plane is given by

$$\Pi_5 = \sum_{\mu=1}^{5} N_5^\mu \exp \mathrm{i}(b^\mu t + \gamma^\mu) \tag{10.23}$$

where forced terms do not appear due to the choice made for the arbitrary constant α. In order to have the position of the equator of Jupiter with respect to the actual mean orbit of the planet, we have to consider the spherical triangle shown in figure 10.3, which may be solved with respect to these parameters. Except for third-degree terms, wc have:

$$\widetilde{I}'^2 = \widetilde{I}^2 + I_0^2 - 2\widetilde{I} I_0 \cos(\widetilde{\Omega} - \Omega_0)$$

$$I_0 \sin(\widetilde{\Omega} - \Omega_0) = \widetilde{I}' \sin(\widetilde{\Omega}' - \widetilde{\Omega}).$$

Therefore

$$\widetilde{I}' \exp \mathrm{i}\widetilde{\Omega}' = \widetilde{I} \exp \mathrm{i}\widetilde{\Omega} - I_0 \exp \mathrm{i}\Omega_0$$

or

$$\Pi_5' = \Pi_5 - \Pi_0. \tag{10.24}$$

It then follows

$$\Pi_5' = \sum_{\mu=1}^{5} N_5^\mu \exp \mathrm{i}(b^\mu t + \gamma^\mu) - \mathrm{i} t S_0 \exp \mathrm{i}\sigma.$$

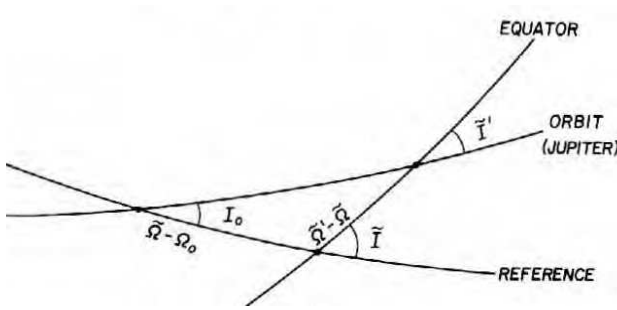

Fig. 10.3.

The numerical values of N_5^μ and S_0 are very small compared to N_5^5. Let then the parameters

$$\theta^\mu = \frac{N_5^\mu}{N_5^5} \qquad (\mu \neq 5), \qquad\qquad \theta^5 = \frac{tS_0}{N_5^5},$$

and the angle

$$\widetilde{\Omega}'' = \widetilde{\Omega}' - b^5 t - \gamma^5$$

be introduced. We have

$$\widetilde{I}' \exp \mathfrak{i}\widetilde{\Omega}'' = \Pi_5' \exp \mathfrak{i}(-b^5 t - \gamma^5);$$

hence

$$\widetilde{I}' \exp \mathfrak{i}\widetilde{\Omega}'' = N_5^5 \big(1 + \sum_1^5 \theta^\mu \exp \mathfrak{i}\beta^\mu\big) \tag{10.25}$$

where

$$\begin{aligned} \beta^\mu &= (b^\mu - b^5)t + (\gamma^\mu - \gamma^5) \qquad (\mu \neq 5), \\ \beta^5 &= \sigma - b^5 t - \gamma^5 - \frac{\pi}{2}. \end{aligned}$$

The absolute value of the summation in equation (10.25) is very small when compared to unity. Equation (10.25) may be easily solved for the unknowns $\widetilde{I}'$ and $\widetilde{\Omega}''$. The result is

$$\widetilde{I}' = N_5^5 \big(1 + \sum_{\mu=1}^5 \theta^\mu \cos\beta^\mu\big)$$

$$\widetilde{\Omega}'' = \sum_{\mu=1}^5 \theta^\mu \sin\beta^\mu$$

and

$$\widetilde{\Omega}' = b^5 t + \gamma^5 + \sum_{\mu=1}^5 \theta^\mu \sin\beta^\mu.$$

$\widetilde{\Omega}'$ is the motion of the jovian equinox. It has a retrograde ($b^5 < 0$) linear part – the jovian *luni-solar precession* – of 2.5 arcseconds per year. The *nutation* terms $\theta^\mu \sin\beta^\mu$ ($\mu \leq 4$) are periodic and their periods are the periods of revolution of the satellite nodes in a reference system affected by the jovian luni-solar precession. The fifth term: $\theta^5 \sin\beta^5$ is, approximately,

$$-\theta^5 \cos(\sigma - \gamma^5) \approx \frac{tS_0}{N_5^5} \cos(\sigma - \gamma^5)$$

which is the jovian *planetary precession*. If we adopt $N_5^5 = 3.11°$ and $\gamma^5 = 316.4°$ (see Section 11.1), we have +3.0 arcseconds per year. The jovian *general* *precession* in longitude is then direct and very small: just 0.5 arcseconds per year.

The inclination is affected by the *nutation* terms $N_5^\mu \cos\beta^\mu$ ($\mu < 5$). For $\mu = 5$, we have $tS_0 \cos\beta^\mu$, or the linear approximation

$$tS_0 \sin(\sigma - \gamma^5).$$

The corresponding secular variation of the jovian mean obliquity is -2.1 arcseconds per century.

10.6 Periodic Inequalities

We have discussed every part of the disturbing function, except one. The terms that have not been discussed so far are

$$\frac{3}{8}\frac{Gm_0}{a_0}\Big(\frac{a_j}{a_0}\Big)^2 \left(I_j^2 \cos(2\lambda_0 - 2\Omega_j) - 2I_0 I_j \cos(2\lambda_0 - \Omega_i - \Omega_0)\right)$$

of solar origin, and the term

$$\frac{3}{4}\frac{GJ_2}{a_j^3}\left(I_j^2 \cos(2\lambda_j - 2\Omega_j) - 2I_j \widetilde{I} \cos(2\lambda_j - \Omega_j - \widetilde{\Omega})\right)$$

that originates from the planet's force-function. Two other terms of the same order were not considered since they are independent of Ω_j and I_j and thus may not give rise to inequalities of the same order as the considered ones. From Souillart's complement to the force-function (eqn 3.23), we get

$$\frac{Gm_k B_{12}^3}{8}\Re\left((\Pi_1^* - \Pi_2^*)^2 \exp 2\mathrm{i}u\right) + \frac{Gm_k B_{23}^3}{8}\Re\left((\Pi_2^* - \Pi_3^*)^2 \exp 2\mathrm{i}u'\right).$$

These parts add some terms having periodic coefficients to the right-hand side of equation (10.7). They are

$$\mathrm{i}(j)(\Pi_j^* - \Pi_0^*)\exp 2\mathrm{i}\lambda_0 + \mathrm{i}(j,5)(\Pi_j^* - \Pi_5^*)\exp 2\mathrm{i}\lambda_j$$

from the first two parts, and from Souillart's complement to the force-function:

$$\frac{\mathrm{i}Gm_2}{4n_1a_1^2}B_{12}^3(\Pi_1^* - \Pi_2^*)\exp 2\mathrm{i}u,$$

$$\frac{\mathrm{i}Gm_3}{4n_2a_2^2}B_{23}^3(\Pi_2^* - \Pi_3^*)\exp 2\mathrm{i}u' - \frac{\mathrm{i}Gm_1}{4n_2a_2^2}B_{12}^3(\Pi_1^* - \Pi_2^*)\exp 2\mathrm{i}u$$

and

$$\frac{-\mathrm{i}Gm_2}{4n_3a_3^2}B_{23}^3(\Pi_2^* - \Pi_3^*)\exp 2\mathrm{i}u',$$

for the first, second and third satellites, respectively.

In the actual calculations Π_j^* are substituted by the general solutions of the homogeneous part as given by equations (10.12). Also, the left-hand side of the equations are not considered in its complete form as in equations (10.11) but as given by the separated approximate equations (10.10). As an example, for the first satellite, we have

$$\frac{d\Pi_1}{dt} - \mathfrak{i}(1,1)\Pi_1 = \mathfrak{i}(1)(\Pi_1^* - \Pi_0^*)\exp 2\mathfrak{i}\lambda_0$$

$$+\mathfrak{i}(1,5)(\Pi_1^* - \Pi_5^*)\exp 2\mathfrak{i}\lambda_1 + \frac{\mathfrak{i}Gm_2}{4n_1a_1^2}B_{12}^3(\Pi_1^* - \Pi_2^*)\exp 2\mathfrak{i}u$$

and the resulting periodic inequalities

$$\begin{aligned}\delta\Pi_1 \; = \; & \sum_\mu \frac{(1)N_1^\mu}{2n_0 - b^\mu - (1,1)}\exp\mathfrak{i}(2\lambda_0 - b^\mu t - \gamma^\mu) \\ & + \sum_\mu \frac{(1,5)(N_1^\mu - N_5^\mu)}{2n_1 - b^\mu - (1,1)}\exp\mathfrak{i}(2\lambda_1 - b^\mu t - \gamma^\mu) \\ & - \sum_\mu \frac{Gm_2B_{12}^3}{4n_1a_1^2}\frac{N_1^\mu - N_2^\mu}{2\mathfrak{m} + b^\mu + (1,1)}\exp\mathfrak{i}(2u - b^\mu t - \gamma^\mu).\end{aligned}$$

There are also variations related to the motion of the orbital plane of Jupiter, i.e., to the variation of Π_0:

$$\frac{K_0(1)}{2n_0 - (1,1)}\left(\frac{1}{2n_0 - (1.1)} - \mathfrak{i}t\right)\exp 2\mathfrak{i}\lambda_0.$$

These contributions are very small. Among the periodic oscillations described above, we consider just those that are increased by the smallness of the denominator. The terms of solar origin increase in importance with j and μ. We select:

$$\begin{aligned}\delta\Pi_1 &= 4.6\times 10^{-6}\exp\mathfrak{i}(2\lambda_0 - b^5t - \gamma_5) \\ \delta\Pi_2 &= 13.8\times 10^{-6}\exp\mathfrak{i}(2\lambda_0 - b^5t - \gamma_5) \\ \delta\Pi_3 &= 31.0\times 10^{-6}\exp\mathfrak{i}(2\lambda_0 - b^5t - \gamma_5) \\ \delta\Pi_4 &= 66.5\times 10^{-6}\exp\mathfrak{i}(2\lambda_0 - b^5t - \gamma_5) + 6.0\times 10^{-6}\exp\mathfrak{i}(2\lambda_0 - b^4t - \gamma_4).\end{aligned}$$

Souillart's terms are of the order of 10^{-7} and are not calculated explicitly here.

The corresponding inequalities in the latitude of the satellites are calculated in the same way as in Section 10.4. For example

$$\delta\psi_1 = \delta\phi_1' = 4.6\times 10^{-6}\sin(\lambda_1 - 2\lambda_0 + b^5t + \gamma_5)$$

and so on.

References and Notes

- 10.3
  The motion of the orbital plane of Jupiter is from
  D.Brouwer and A.J.J. van Woerkom: 1950, "The Secular Variations of the Orbital Elements of the Principal Planets", *Astronomical Papers of the American Ephemeris*, Vol. XIII, Pt. II.
  The results of Souillart and Tisserand are given in
  C.Souillart: 1894, "Sur certains termes complémentaires des Expressions des Latitudes dans la Théorie des Satellites de Jupiter", *Bulletin Astronomique* 11, 145-158.
  F.Tisserand: 1894 "Note au sujet du Mémoire Précedent", *Bulletin Astronomique* 11, 159.

- 10.4
  In Table 10.6, we refer to the errors taken from the original publications. The comparison shows that the standard errors in de Sitter's results are underestimated.

- 10.5
  The planetary precession may be calculated with greater accuracy by using the motion of the orbital plane of Jupiter from
  P.Bretagnon: 1978, "Sur une Solution Globale du Mouvement des Planètes" *Thèse de Doctorat*, Université Pierre et Marie Curie, Paris.

11

Elements and Physical Parameters

11.1 Physical Parameters

The theory of motion of the Galilean satellites of Jupiter involves more than thirty physical parameters and integration constants, which are to be determined from observations. There are 6 integration constants for each satellite orbit, 2 for the motion of the pole of Jupiter and several physical parameters. These physical parameters are associated with disturbing forces. Until recently, the determination of the masses of the satellites and the second harmonic of Jupiter's potential, was done simultaneously with the orbital elements. Other parameter, the fourth harmonic, was determined from the motion of the node of the innermost Jovian satellite: Jupiter V (Amalthea). de Sitter, using a known value of J_2 and the formula derived by H.Struve determined J_4. More recently, Brouwer and Clemence obtained

$$23269 J_2 - 8121.6 J_2^2 - 9024.0 J_4 = 346.53 \pm 0.14. \tag{11.1}$$

With modern space probes flying in the vicinity of Jupiter, the main parameters can be determined independently. The first effort was made at the Jet Propulsion Laboratory (JPL), Pasadena, California, by analyzing the Doppler shift of the signals emitted by the spacecrafts Pioneer 10 and Pioneer 11 when they were near Jupiter. These results together with the classical results obtained by Laplace, Sampson and de Sitter are shown in Table 11.1. The last column of Table 11.1 shows the values recommended by the Sixteenth General Assembly of the International Astronomical Union to be used in the preparation of ephemerides; these IAU values have been adopted in this book.

It is worth noting that the JPL determination of J_4 agrees completely with the values expected by using equation (11.1). Further, the attempts to determine J_3, J_6 and J_{22} have been unsuccessful within 1×10^{-6} for J_{22}, 1×10^{-5} for J_3 and 5×10^{-5} for J_6.

Other physical parameters are related to Jupiter: size, rotation, mass, moments of inertia, etc. For the equatorial radius of Jupiter, the value recommended by IAU is b =71398 km, which is based on the Pioneer 10 and 11

Table 11.1. Satellite masses and Jupiter's J_2 and J_4

Parameter	Laplace	Sampson	de Sitter	JPL	IAU
Masses $\times 10^6 M_J$					
m_1	17	45.0	38.1±4.5	48.84±.22	47.0
m_2	23	25.4	24.8±1	25.23±.25	25.6
m_3	88	79.9	81.7±1.5	78.03±.30	78.4
m_4	42	45.0	50.9±6	56.61±.19	56.0
Harmonics $\times 10^4$					
J_2	219	148.5	145.3±2	147.33±.04	147.5
J_4			8.9	-5.87±.07	-5.8

studies. The mass of Jupiter with respect to the solar mass is well known and we adopted the reciprocal of 1047.572 (IAU recommended value 1047.355 refers to the whole Jovian system). The rotation of Jupiter is a parameter difficult to be determined: we can observe only the upper atmosphere of the planet and the observed rotation depends on the latitude. We have adopted the radioastronomical determination $P = 0.41354$ d of the period of Jupiter as it is believed that the radio signals originate from deeper regions and may be related to the actual body of Jupiter. The moment of inertia of Jupiter is ill determined and we have adopted C=0.26.

The position of the pole of Jupiter, determined from the motion of Pioneer 10 is

$$\begin{aligned} \alpha &= 267.998 \pm 0.016^\circ \\ \delta &= 64.504 \pm 0.004^\circ \end{aligned}$$

on December 3, 1974. The transformation in $\widetilde{\Omega}'$ and $\widetilde{I}'$ gives

$$\begin{aligned} \widetilde{I}' &= 3.11 \pm 0.03^\circ \\ \widetilde{\Omega}' &= 316.4 \pm 0.5^\circ. \end{aligned}$$

Precision is lost in the transformation because of uncertainties in figures giving the position of Jupiter's orbit derived from the theory of Brouwer and Van Woerkom. This determination is compatible with Sampson's values for 1900.0:

$$\begin{aligned} \widetilde{I}' &= 3.1035^\circ \\ \widetilde{\Omega}' &= 316.051^\circ. \end{aligned}$$

In this book we have adopted for the epoch 1950.0 the values

$$\begin{aligned} \widetilde{I}' &= 3.103^\circ \\ \widetilde{\Omega}' &= 316.06^\circ. \end{aligned}$$

11.2 Sampson's Orbital Elements

The orbital elements determined by Sampson were obtained from the analysis of several series of photometric records of eclipses of the satellites made at the Harvard College Observatory from 1878 to 1903. These observations were supplemented by visual observations of eclipses collected by Delambre, for the determination of two secular motions, and also supplemented by the results of Damoiseau for the determination of the mean motions.

Sampson first determined the mean longitudes at a fixed epoch (1890 January 1.0). The observations were reduced twice, independently, at Harvard and Durham. The average longitude for the three inner satellites were found to be

$$\lambda_1 = 242.9671^\circ \qquad \lambda_2 = 59.6381^\circ \qquad \lambda_3 = 57.9727^\circ$$

and thus

$$\lambda_1 - 3\lambda_2 + 2\lambda_3 = 180^\circ - 0.0018^\circ.$$

Sampson assumed the sum to be equal to 180° and to fit such condition, he introduced the corrections $+0.0003^\circ$, -0.0003° and $+0.0003^\circ$, respectively. Thus,

$$\lambda_1 = 242.9674^\circ \qquad \lambda_2 = 59.6378^\circ \qquad \lambda_3 = 57.9730^\circ.$$

The corresponding values from Damoiseau's tables are

$$\lambda_1 = 243.0160^\circ \qquad \lambda_2 = 59.6578^\circ \qquad \lambda_3 = 57.9803^\circ.$$

The comparison between these values indicates corrections to Damoiseau's values by

$$-\,0.0486^\circ \qquad\qquad -\,0.0200^\circ \qquad\qquad -\,0.0073^\circ$$

or corrections to Damoiseau's daily tropic motions yielding the values

$$\begin{gathered} 203.488\,992\,435^\circ/\text{day} \\ 101.374\,761\,672^\circ/\text{day} \\ 50.317\,646\,290^\circ/\text{day}. \end{gathered}$$

For Jupiter IV (Callisto), processing in exactly the same way, except for libration corrections, Sampson found the correction to Damoiseau's tables to be $+0.0426^\circ$. However, it was not clear that this should be attributed to erroneous mean motion or not. Delambre's collection of eclipses, when rediscussed, showed that at epoch 1788.79 Damoiseau's mean longitude was in error by $+0.0352^\circ$. This was accepted by Sampson and total correction was then $+0.0074^\circ$ in 36966 days. The corrected daily tropic motion resulted is

$$21.571\,109\,630^\circ/\text{day}.$$

In order to have sidereal mean motions, it is sufficient to consider the precession. The resulting values are

$$\begin{aligned} n_1 &= 203.488\,954\,208^\circ/\text{day} \\ n_2 &= 101.374\,723\,445^\circ/\text{day} \\ n_3 &= 50.317\,608\,063^\circ/\text{day} \\ n_4 &= 21.571\,071\,403^\circ/\text{day}, \end{aligned}$$

which have been adopted by Sampson.

The position of perijoves and nodes at the epoch JD 2415020.0 (1900 January 0.5) determined by Sampson are given in Table 11.2.

Table 11.2. Perijoves and Nodes (1900 Jan. 0.5)

Satellite	Perijove	Node
1	265.719°	33.299°
2	196.534°	290.550°
3	340.679°	320.705°
4	283.258°	7.331°

The other elements determined by Sampson have been considered in details in preceding chapters.

11.3 Sampson's Time Scale

The nominal time scale in Sampson's tables is the mean solar day which is not uniform. The theory of Sampson as well as every mechanical theory defines a proper time scale, uniform except for the errors of the theory itself. This scale is directly related to the observations used for sake of obtaining the integration constants of the theory, i.e., the elements.

The first studies on Sampson's time scale (t_S) were made by Rodrigues (1970) assuming that the shift of t_S leads to a systematic component in the observed positions. His results have been confirmed by further studies. Some recent results are shown in Table 11.3.

In these results, the shift of Sampson's time scale is a weighted mean of the least squares solution for the shift of the individual time scale of each satellite. The results show that a significant shift in Sampson's time scale exists. A very simplified model may help us to have a better understanding of what happens.

Let us suppose a simple phenomenon whose state is given by a measurable parameter θ and assume that a theory exists showing that θ depends on the uniform time through a linear function $L(t)$:

$$\theta = f\big(L(t)\big). \tag{11.2}$$

If the mean motion L is determined from two sets of measurements made at the mean epochs t_a and t_0, it is possible to evaluate the effects of using a non-uniform time scale (UT) as it happened in the derivation of the mean motions by Sampson. He used the law

$$\theta = f\big(L(s)\big) \tag{11.3}$$

where s is the same instant as t but in the UT-scale.

Table 11.3. Values of Δt

Epoch	t_S − E.T. (min)	Observatory/Observer
1913.5	0.29 ± 0.03	Cape/*
1914.6	0.25 ± 0.02	Cape/*
1915.7	0.30 ± 0.03	Cape/*
1916.8	0.24 ± 0.02	Greenwich/**
1919.0	0.29 ± 0.03	Greenwich/**
1922.3	0.24 ± 0.04	Leiden/W.H.van den Bos
1924.6	0.22 ± 0.04	Cape/*
1927.5	0.36 ± 0.04	Johannesburg/H.L.Alden
1928.7	0.49 ± 0.05	Johannesburg/H.L.Alden
1934.4	0.28 ± 0.08	Bucharest/G.Petrescu
1968.2	0.41 ± 0.04	Leander-McCormick/D.Pascu
1973.7	0.67 ± 0.03	U.S.N.O. Washington/D.Pascu
1974.7	0.26 ± 0.04	U.S.N.O. Washington/D.Pascu

* J.Lunt, J.W.Jackson, R.Woodgate and G.Duncan
** M.Jones, C.Davidson, P. Melotte and E.Martin

The correct determination of the mean motion is given by

$$\dot{L}_E = \frac{g(\theta_0) - g(\theta_a)}{t_0 - t_a} \tag{11.4}$$

where $g = f^{-1}$. The actual calculations have been made with the improper law (11.3):

$$\dot{L}_U = \frac{g(\theta_0) - g(\theta_a)}{s_0 - s_a}. \tag{11.5}$$

The relationship between these two mean motions may be written as

$$\lambda = \frac{\dot{L}_E}{\dot{L}_U} = 1 - \frac{d_0 - d_a}{t_0 - t_a}$$

where d_0 and d_a are the differences ET−UT in the considered epochs:

$$d_0 = t_0 - s_0 \qquad\qquad d_a - t_a - s_a.$$

Let this theory be used for a new set of measurements made at the mean epoch t. Since the UT is not uniform, the improper law will not reproduce the measurements. Let Δs be the correction of the time scale required to get the observed result with the improper law. Then

$$\theta = f\big(L_U(s + \Delta s)\big)$$

where L_U is the linear function L with $\dot{L}_U$ as mean motion. Therefore

$$g(\theta) = L_U(s + \Delta s) = \dot{L}_U(s + \Delta s - s_0) + L_U(s_0).$$

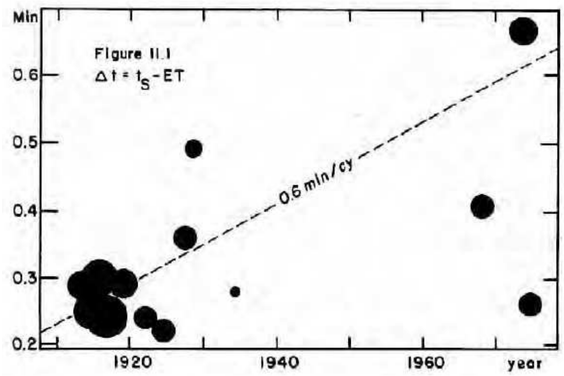

Fig. 11.1.

We assume that the correct law (11.2) would reproduce exactly the observations; thus

$$g(\theta) = L_E(t) = \dot{L}_E(t - t_0) + L_E(t_0).$$

We also assume that both laws reproduce exactly the measurements at the mean epoch t_0,

$$g(\theta_0) = L_U(s_0) = L_E(t_0).$$

Then

$$\dot{L}_U(s + \Delta s - s_0) = \dot{L}_E(t - t_0)$$

and after some calculations, we have

$$\Delta t = (\lambda - 1)(t - t_0) - d_0 \tag{11.6}$$

where $\Delta t = t_s - \mathrm{ET}$.

Let us now apply this simple model to Sampson's tables. The old data used by Sampson were longitudes provided by Damoiseau's tables, which depend on eclipses observed from the end of the seventeenth century to about 1830. The current values of ET−UT adopted for that epoch and for the mean epoch of the Harvard eclipses are

$$t_a = 1750.0 \qquad d_a = 0 \text{ s}$$
$$t_0 = 1890.0 \qquad d_0 = -7 \text{ s}$$

and thus we have

$$\lambda - 1 = 0.05 \text{ s/yr}.$$

The adopted values for ET−UT were obtained by D.Brouwer who compared observations of the Moon to the theory of E.W.Brown. Brown's adopted tidal acceleration of the mean longitude of the Moon was $-11.22''T^2$ (where T is in centuries). Van Flandern claims that this acceleration needs a correction $-10''T^2$. The Moon moves one arcsecond per 1.82144 seconds and the corresponding correction in Brouwer's determination of ET is $+18.21T^2$. He then suggests the correction

$$0.15 - 2.55(T - 19.63) + 18.21(T - 19.63)^2 \text{ seconds.}$$

If this correction is adopted we have

$$d_a^* = 88 \text{ s} \qquad\qquad d_0^* = 5 \text{ s}$$

and

$$\lambda^* - 1 = 0.6 \text{ s/yr.}$$

On adopting the time-scale correction of 0.6 minutes per century (0.36 s/yr), we have the sidereal mean motions

$$\begin{array}{rl} 203.488\,956\,4 & \pm\ 0.0000004°/\text{day} \\ 101.374\,724\,5 & \pm\qquad\quad 2 \\ 50.317\,608\,6 & \pm\qquad\quad 1 \\ 21.571\,071\,64 & \pm\qquad\quad 05, \end{array}$$

which have been used throughout this book[1].

11.4 On Accelerations

The time scale of Sampson's theory may not explain the observed deviations in the longitude of the satellites. There are errors in other elements, viz. positions of perijoves and nodes, and there are errors in the coefficients of important long period inequalities. Besides, neglected long period inequalities and accelerations may exist. The extension of the model considered in previous section to the case where accelerations exist is very easy. Instead of equation (11.2) suppose that the state of the phenomenon is given by

$$\theta = f\big(Q(t)\big) \qquad\qquad Q = a + bt + \frac{1}{2}ct^2. \tag{11.7}$$

The mean motions derived by means of equations (11.4) and (11.5) from the sets of measurements made at the mean epochs t_a and t_0, respectively, are

$$\dot{L}_E = \lambda \dot{L}_U = b + \frac{1}{2}c(t_a + t_0)$$

where $\dot{L}_E$ is the average of $\dot{Q}$ in the time interval. For the set of measurements made at the mean epoch t, we have

$$Q(t) = L_U(s + \Delta s),$$

[1] Hence, $\mathfrak{m} = 0.739\ 507°/\text{day} = 0.01290\text{d}^{-1}$

which defines Δs. Similarly, for the mean epoch t_0, we have

$$Q(t_0) = L_U(s_0)$$

since $\Delta s = 0$ at epoch t_0. Thus

$$g(\theta) = L_U(s_0) + \dot{L}_E(t - t_0) + \dot{L}_U\big(\Delta t + d_0 + (1 - \lambda)(t - t_0)\big)$$

and

$$g(\theta) = Q(t_0) + \big(b + \frac{1}{2}c(t_0 + t)\big)(t - t_0).$$

Neglecting higher-order corrections in these relations, it follows

$$\frac{c}{2b} = \frac{\Delta t + d_0 - (\lambda - 1)(t - t_0)}{(t - t_a)(t - t_0)}.$$

From the data given in the previous section we have $c/b = +7 \times 10^{-9}\mathrm{cy}^{-1}$ were we used Brouwer's results. Using Van Flandern scale ET*, we have $c/b = -5 \times 10^{-9}\mathrm{cy}^{-1}$. These results need some discussions.

Data listed in Table 11.3 are averages of the evolution of the satellites. Accelerations are expected because of secular variations in the orbit of Jupiter and, in this case, the outermost satellites would be more affected than the inner satellites. Accelerations are also expected because of tidal friction and in this case only the inner satellites would be affected. But observational consideration of each satellite separately is impaired by the existence of important long-period residuals (which are more or less averaged when we consider a joint time-scale correction). At the end, the model used is very simplified. The results only serve to give an idea of the observational uncertainty still existing before acceleration is detected.

When tidal friction is neglected, the most probable source of acceleration in the motion of a satellite is the secular variation of eccentricity of the orbit of the central planet around the Sun. The quadratic term in the longitude of the epoch is given by the equation

$$\frac{d\varepsilon^{\mathrm{I}}}{dt} = -\frac{3Gm_0e_0^2}{2na_0^3} \simeq -\frac{3n_0^2e_0^2}{2n}$$

where $e_0 = \mathrm{const} + \dot{e}_0\mathrm{t}$. After integration, the time dependent part of e_0 gives

$$\frac{1}{2}ct^2 = \delta_2\varepsilon^{\mathrm{I}} = -\frac{3n_0^2}{2n}e_0\dot{e}_0t^2$$

which is very small. After Brouwer and Van Woerkom $\dot{e}_0 = 1.59 \times 10^{-6}\ \mathrm{yr}^{-1}$. Therefore, for the Galilean satellites, $nc = 4.85 \times 10^{-11}\ \mathrm{d}^{-2}\mathrm{cy}^{-1}$. The maximum rate c/b is that of Jupiter IV (Callisto): $-3.4 \times 10^{-10}\mathrm{cy}^{-1}$.

Tidal effects are important sources of evolutionary inequalities in a system of satellites. The classical formula is

$$\frac{\dot{a}}{a} = 3\left(\frac{G}{Ma^{13}}\right)^{1/2} mk_2 b^5 \sin 2\epsilon$$

or

$$\frac{\dot{n}}{n} = -\frac{3\dot{a}}{2a} = -\frac{9mnk_2}{2a^5}\sin 2\epsilon.$$

In these formulae k_2 is the tidal Love number and ϵ is the tidal lag angle: the angle between the maximum tidal bulge and the planet-satellite line of centres. For the innermost satellite, we have

$$\frac{\dot{n}}{n} = -0.08 k_2 \sin 2\epsilon \ \mathrm{cy}^{-1}$$

where the fact that it will redistribute part of its effects to other satellites because of libration was already considered. For almost circular orbits, the sign of c is governed by the sign of $r-n$ which is positive. Therefore, whatever is $k_2 \sin 2\epsilon$, the acceleration is negative.

At last it must be kept in mind that important long-period residuals still exist in Sampson's tables. These residuals avoid a meaningful study of the accelerations. Nevertheless, new campaigns of observations and the reconsideration of old data progress. We are confident that a better knowledge of long-period terms will be soon available and will allow to consider the residual accelerations with more rigour and to obtain some results.

The values obtained by de Sitter 50 years ago of the order of 10^{-8} cy^{-1} are not acceptable. Their corrections to get a uniform time scale were based on a uniform deceleration of the Earth whereas, in fact, the Earth's deceleration is in no way uniform at all.

11.5 Elements of de Sitter

The determination of the elements by de Sitter, who considered all available observations made before 1928, was much more eclectic than that of Sampson.

(a) Photographic observations: de Sitter considered about ten series of photographic observations made at the Observatories in Helsingfors, Cape, Pulkovo, Greenwich, Leiden and Johannesburg in the period 1891-1927. The telescopes used had focal lengths ranging from 3.43m (Carte du Ciel telescopes at Helsingfors and Pulkovo) to 10.9m (Yale-Columbia Southern Station telescope at Johannesburg). The precision of these series were studied by de Sitter, who determined the standard errors involved for one coordinate (averages of 6 exposures measured in two positions). The results are shown in Table 11.4. These results have not yet been confirmed by more recent analysis of the observational data.

(b) Micrometric observations: Several series of micrometric observations made between 1903 and 1909 at Washington and Berlin were considered.

Table 11.4. Standard Errors

Telescope	Focal Length	Standard Error
Carte du Ciel	3.4 m	0.08″
Leiden	5.2	0.05″
Cape and Greenwich	6.8	0.05″
Johannesburg	10.9	0.03″

(c) Heliometric observations: de Sitter considered the series of observations made with the Cape heliometer (focal length 2.5m) by Gill and Finlay in 1891 and by Cookson in 1901 and 1902.

(d) Observations of eclipses and phenomena: de Sitter also considered parts of the old collection of eclipses used by Wargentin, Delambre and Damoiseau, and the photometric series of Harvard as well as the very detailed observations of phenomena made by Innes and Wood at Johannesburg and discussed by Brouwer.

From the point of view of their intrinsic precisions, it is interesting to compare these different kinds of observations. Table 11.5 shows de Sitter's estimates of the standard errors converted in time (seconds).

Table 11.5. Standard Errors in seconds

Satellite	I	II	III	IV
Eclipses (Harvard)	10	16	17	32
Heliometer (D.Gill)	9	12	15	19
Photographs (Johannesburg)	5	7	9	12

Considering all data listed above, de Sitter deduced elements. Table 11.6 lists the mean motions and the positions of the perijoves and nodes at the same epoch of Sampson's elements. Other elements were considered in details in the preceding chapters.

Table 11.6. Mean Motions, Perijoves and Nodes

Satellite	Mean Motion (°/d)	Perijove	Node
1	$203.4889\ 9636\pm20\times10^{-8}$	$75\pm32°$	$64.7\pm3.8°$
2	$101.3747\ 6336\pm17\times10^{-8}$	149.5 ± 6.6	292.81 ± 0.16
3	$50.3176\ 4706\pm15\times10^{-8}$	345.6 ± 1.2	319.83 ± 0.46
4	$21.5711\ 1041\pm20\times10^{-8}$	282.79 ± 0.10	12.79 ± 0.31

Although de Sitter's results have one more digit than allowed by the magnitude of the standard errors, they have less digits than Sampson's results. The results given by Sampson have many meaningless digits; they have been kept in this book in order to reproduce exactly the data used by Sampson in his theory and in his tables.

11.6 Other Observational Data

To determine the physical parameters of the satellites, Sampson, as well as de Sitter, determined from the observations the amplitudes of several inequalities and some daily motions. The quantities that were determined by them from the observations are:

(a) The coefficient $2\overline{B}_1$ of the induced equation of the centre in the longitude of Io (see Section 6.1);

(b) The coefficient $2\overline{B}_2$ of the induced equation of the centre in the longitude of Europa (see Section 6.1);

(c) The coefficient $2M_3^4$ of the free oscillation in the longitude of Ganymede whose argument is $\lambda_3 - g^4t - \beta^4$ (see Sections 5.6 and 7.7);

(d) The daily motion g^4 of the proper apsis of Callisto (see Sections 5.6 and 7.7);

(e) The daily motion b^2 of the proper node of Europa (see Section 10.2).

Their results are shown in Table 11.7 together with the results obtained in this book using the IAU recommended values of the physical parameters.

de Sitter also made determinations of other proper apsides and nodes (see Tables 5.4 and 10.2) as well as the amplitude of some other inequalities:

(f) The coefficient N_3^4 of the free oscillation in the latitude of Ganymede whose argument is $\lambda_3 - b^4t - \gamma_4$ (see Section 10.4), and

(g) The coefficient $2\overline{B}_3$ of the induced equation of the centre in the longitude of Ganymede (see Section 6.1).

Table 11.7. Comparison of some quantities (in units 10^{-5}

Quantity	Computed	Deduced from Observations	
		Sampson	de Sitter
$2\overline{B}_1$	827.0	823.0	821.5±3.4
$-2\overline{B}_2$	1852.6	1867.8	1866.5±4.8
$2M_3^4$	145	128.8	134.3±12.6
g^4	3.21	3.24	3.29±0.05
b^2	58.0	56.7	56.9±0.7
N_3^4	66.1		50.9±4.0
$2\overline{B}_3$	121.0		112±15

11.7 Mutual Events Results

A concerted campaign and much international cooperation provided a large collection of observations of mutual phenomena of the Galilean satellites during the favorable passage of the Earth through the plane of Jupiter's equator in 1973. Aksnes and Franklin made an analysis of 91 mutual eclipses and occultations that occurred from June to December 1973. They obtained four different least-squares solutions for longitude corrections and two different least-squares solutions for latitude corrections.

The best solutions for the proper elements at the mean epoch of the observations JD 2441 920.5 (1973.65) are shown in Table 11.8.

Table 11.8. Mutual events results

Satellite	Proper Perijove	Proper Eccentricity	Proper Node	Proper Inclination
1	$328 \pm 17°$	$33 \pm 9 \times 10^{-5}$	$117 \pm 8°$	$73 \pm 23 \times 10^{-5}$
2	43 ± 23	15 ± 4	136.4 ± 0.5	824 ± 16
3	174.7 ± 0.9	152 ± 3	128.2 ± 0.8	384 ± 19
4	332.88 ± 0.09	730 ± 4	321.3 ± 0.4	377 ± 23

They also determined from the observations the coefficients of the two largest inequalities in longitude:

$$\overline{B}_1 = (416 \pm 10) \times 10^{-5}$$

$$-\overline{B}_2 = (928 \pm 5) \times 10^{-5}.$$

If the values in Table 11.8 are compared to Sampson's tables, some important disagreements arise: (a) the node of Io is 90° far from the value predicted in Sampson's tables for the mean epoch of the observations (26.8°); (b) the perijoves of Io and Europa have shifts of 135° and 20°, respectively; (c) remaining perijoves and nodes have shifts that are smaller but important because the corresponding eccentricities and inclinations are large.

The discrepancies arise mainly from some unexpected bad values for some characteristic roots in Sampson's tables. Table 11.9 shows the motion of the proper perijoves and nodes deduced from comparison of Sampson's determinations of the perijoves and nodes at 1900 Jan. 0.5 to mutual event results. Motions obtained with de Sitter's determinations instead of Sampson's are shown in brackets. Table 11.9 also shows the characteristic roots in Sampson's tables, in Lieske's ephemerides E-2 (see Section 11.8) and the values obtained in this book. The bad value of the motion of Jupiter II (Europa) obtained with the approximate theory given in this book is an example of the importance of high-order terms in the theory of the motion of the Galilean satellites.

Table 11.9. Motion of proper perijoves and nodes (in units $10^{-6}\mathrm{d}^{-1}$

	Satellite	Deduced from Tables 11.8 and 11.2		Sampson's Tables	Lieske's Ephemeris E-2	Tables 7.2 and 10.2
Perijove						
	I	2840	[2733]	2756	2810	2731
	II	835	[865]	822	811	700
	III	126		121	124	130
	IV	32.2		32.4	32.1	32.0
Node						
	I	-2280	[-2302]	-2340	-2318	-2331
	II	-567	[-569]	-571	-569	-580
	III	-125		-123	-125	-126
	IV	-29.9		-32	-30.7	-30.8

11.8 Lieske's Ephemeris

Recently, at the Jet Propulsion Laboratory, Lieske developed analytic expressions for the positions and partial derivatives of the satellites utilizing the same method as Sampson and made a preliminary evaluation of the constants employed in the new theory in order to best fit observations. He started with the analysis of the photometric eclipse observations made at Harvard during the years 1878-1903 as well as the very few photometric eclipse observations of half-brightness made since then. A set of parameters (called E-l) was derived by iteratively fitting and re-fitting the data until the solutions converged. As noted before by Aksnes and Franklin, some parameters (viz. the nodes and apsides) require large corrections. As in Sampson's tables, the orbit of Jupiter IV (Callisto) is derived from few observations (31 in the period 1878-1903 and 7 after 1954).

A second set of parameters (called E-2) was derived in the same way, fitting also the mutual events data, as well as 2964 photographic observations from 1967 to 1978. The photographic observations were obtained with equal telescopes at the U.S. Naval Observatory by D. Pascu, and at the Leander McCormick Observatory by D. Pascu, P. Ianna and P. Seitzer. The mean motions, proper elements and longitudes at the epoch JD 2443 000.5 (1976 August 10.0 ET) for E-2 ephemeris are shown in Tables 11.10 and 11.11.

Table 11.10. Metric Elements in Ephemeris E-2

Satellite	Mean Motion (°/d)	Proper Eccentricity	Proper Inclination
I	203.4889 5536 $\pm$ 75 $\times 10^{-8}$	1$\pm$0.4 $\times 10^{-5}$	70$\pm$28 $\times 10^{-5}$
II	101.3747 2456 $\pm$ 59 $\times 10^{-8}$	9$\pm$2	816$\pm$21
III	*	147$\pm$3	324$\pm$16
IV	21.5710 7087$\pm$ 62 $\times 10^{-8}$	733$\pm$3	443$\pm$49

*$(3n_2 - n_1)/2$ =50.3176 0915 $\pm$ 59 $\times 10^{-8}$

Table 11.11. Angular Elements in Ephemeris E-2

Satellite	Longitude at the epoch	Proper Perijove	Proper Node
I	$106.0786 \pm 0.0176°$	$82 \pm 74°$	$308 \pm 18°$
II	175.7338 ± 0.0039	129 ± 16	100 ± 1
III	*	187.6 ± 0.9	119 ± 2
IV	84.4558 ± 0.0049	335.3 ± 0.1	323 ± 2

*$(3\varepsilon_2 - \varepsilon_1)/2$=$180.5614 \pm 0.0048°$

Since the mean motion and longitude at the epoch of Jupiter III (Ganymede) are derived from Laplace theorems, two other independent integration constants are needed. They are the amplitude and phase of the libration. The two amplitude determinations made by Lieske are shown in Table 11.12.

Table 11.12. Amplitude of the Libration

Lieske's Ephemeris	Amplitude (D)	
E-1	$9.7 \pm 3.5 \times 10^{-4}$	$(3'22'')$
E-2	$11.5 \pm 2.2 \times 10^{-4}$	$(3'57'')$

In both determinations, the phase is close to zero at JD 2443000.5 but the standard error of the determined values is very great. The librations in the longitudes of the three inner satellites are

$$\delta\theta_1 = +0.00014 \sin(n_L t + E) \qquad (27'')$$
$$\delta\theta_2 = +0.00032 \sin(n_L t + E) \qquad (65'')$$
$$\delta\theta_3 = +0.00003 \sin(n_L t + E) \qquad (6'')$$

These inequalities have amplitudes of the same order of magnitude as some among the Great Inequalities in Longitude (see Table 7.1). Lieske's value of n_L is $3.03 \times 10^{-3}\mathrm{d}^{-1}$ which corresponds to a libration period of 2074 days.

The work of Lieske is still in progress and the results of this Section are expected to be improved shortly.

Lieske's set of elements E-2 together with the mean motions derived in Section 11.3 have been widely used all along this book.

References and Notes

- 11.1

  Equation (11.1) is from

D.Brouwer and G.M.Clemence: 1961, "Orbits and Masses of Planets and Satellites". In G.Kuiper and B.G.Middlehurst (eds.), *The Solar System*. Vol. III, Univ. Chicago Press, Chicago, pp. 31-94.

IAU recommended values are given in

R.L.Duncombe, W.Fricke, P.K.Seidelmann and G.A.Wilkins: 1977, "Joint Report of the Working Group of lAU Commission 4 on Precession, Planetary Ephemerides, Units and Time-Scales", *Trans. Intern. Astron. Union* XVIB, 56-67.

JPL values are given in

G.W.Null: 1976, "Gravity Field of Jupiter and its Satellites from Pioneer 10 and Pioneer 11 tracking data", *Astron. Journal* 81, 1153-1161.

It is worth to recall that J_2 and J_4 are related to classical parameters J and K through $J_2 = 2J/3$ and $J_4 = -4K/15$. The adopted period of Jupiter is that of System III, that is 9h 55m 29.711s. For the normalized moment of inertia of Jupiter, we adopted $C = 0.26$. Recent Jovian models, which satisfy Pioneer 11 constraints are given by

W.B.Hubbard and W.L.Slattery: 1976, "Internal Structure of Jupiter". In T.Gehrels (ed.), *Jupiter*, Univ. Arizona Press, Tucson, pp. 176-194.

and indicate for C the classical Jeffreys' value $C = 0.25$.

- 11.2

Sampson's elements are discussed at the end of his Theory.

- 11.3

For the years before 1940, the values of ET-UT are the unsmoothed values Δt given by

D.Brouwer: 1952, "A Study of the Changes in the Rate of Rotation of the Earth". *Astron.Journal* 57, 125-146.

For 1968 onwards, they are from current Ephemeris and have been derived from an atomic time scale that has been fitted to the observed values of ET from lunar observations.

Results in Table 11.3 are mostly from

S. Ferraz-Mello and L. R. de Paula: 1976. "Discussion of the Photographic Observations of the Galilean Satellites in the Period 1930-1970", *Astron. Journal* 81, 127-131.

M. Tsuchida, S. Ferraz-Mello and R. Biancale: 1982. "Discussion of the photographic observations of the Galilean satellites in the period 1913-1928", *Astron. Journal* 87, 924-927.

Some results may be compared to those given by

J.E.Arlot: 1975, "A comparison of some Observations of the Galilean Satellites with Sampson's Tables", *Celestial Mechanics* 12, 39-50.

The standard errors in Arlot's papers are in fact probable errors and for making comparisons they must be divided by 0.6745.

The dashed line in Figure 11.1 crosses the mean epoch of Harvard, series at Brouwer's value $t_s - \mathrm{ET} = \mathrm{UT} - \mathrm{ET} = 7$ s.

Recently, Morrison, using the results of his analysis of the time-residuals of transits of Mercury in the period 1677 to 1973, indicated an observed tidal acceleration of the Moon $26''\mathrm{cy}^{-2}$

L.V Morrison: 1977, "Tidal Deceleration of the Earth's Rotation deduced from Astronomical Observations in the period A.D. 1600 to the present", *N.A.O. Techn. Note* 43, Roy. Greenwich Obs., Hailshan.

These results indicate $\lambda - 1 = 0.1$ s/year and the same d_0 as Brouwer.

- 11.4

A recent study of the eclipse observations of Picard and Roemer from 1668 to 1678, using a simplified model, gives $n_1 = 203.488919(\pm 16 \times 10^{-6})$ deg/day:

S.J.Goldstein Jr.: 1975, "On the Secular Change in the Period of Io, 1668-1926", *Astron. Journal* 80, 532-539.

A comparison with modern values leads to $c/b = (7 \pm 3) \times 10^{-8}$ cy^{-1}. If this value is corrected for the deceleration of the Earth's rotation, it is reduced and comes to the same order of its standard error; that is still greater than the expected value.

- 11.5

de Sitter's work is discussed in the Gold Medal lecture

W.de Sitter: 1931, "Jupiter's Galilean Satellites", *Monthly Notices Roy. Astron. Soc.* 91. 706-736.

de Sitter uses always probable errors; thus we transformed them into standard errors in order to compare his results with others. The comparisons indicate that de Sitter's errors are, perhaps, underestimated.

- 11.7

Mutual events results of the 1973 campaign are collected and discussed in

K.Aksnes and F.A.Franklin: 1976, "Mutual Phenomena of the Galilean Satellites in 1973. Final Results from 91 Light Curves", *Astron. Journal*, 81, 464-481.

See also

T.Nakamura: 1976, "Analysis of Mutual Phenomena of Galilean Satellites in 1973", *Publ. Astr. Soc. Japan* 28, 239-257.

- 11.8

Ephemeris E-1 is discussed in

J.H.Lieske: 1978, "Gali1ean Satellites: Analysis of Photometric Eclipses". *Astron. Astrophys.* 65, 83-92.

The work leading to Ephemeris E-2 has not yet been published.

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