

Limites sobre teorias $f(R)$

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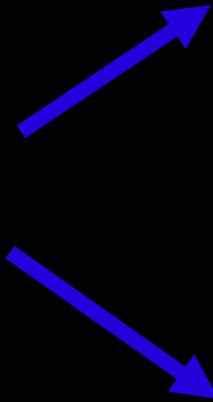
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Teorias $f(R)$



**Regime de curvatura "fraca"
(ex.: aceleração do universo)**

Frolov (2008)



**Regime de curvatura "forte"
(ex.: estrelas de nêutrons,
buracos negros, "very early universe")**

Como limitar a forma de $f(R)$?

a) Construindo uma $f(R)$ que satisfaça condições “razoáveis” (campo fraco, estabilidade, geração de estrutura...), numa região do espaço de parâmetros.

$$f(R) = R + \lambda R_0 \left(\left(1 + \frac{R^2}{R_0^2} \right)^{-n} - 1 \right)$$

Starobinski (2007)

$$f(R) = -m^2 \frac{c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1}$$

Hu e Sawicki (2007)

$$F(R) = \frac{R}{2} + \frac{1}{2a} \log [\cosh(aR - b)] + A,$$

Appleby and Battye (2007)

b) Impondo condições gerais que forneçam vínculos sobre a $f(R)$ (ou relações entre a $f(R)$ e suas derivadas), válidos para qualquer $f(R)$.

Exemplos:

Evolução cosmológica “certa” : inflação (e/ou ricochete) - radiação - matéria - expansão acelerada.

- **Campo fraco.**
- **Geração de estrutura.**
- **“Boa propagação” (ausência de birrefringência e *shocks* no vácuo) .**
- **Existência de buracos negros, estrelas compactas.**
- **. . . .**

Teoria $f(R)$ (versão métrica)

$$S_{met} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R)$$

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - [\nabla_\mu \nabla_\nu - g_{\mu\nu}\square] f'(R) = \kappa T_{\mu\nu}$$

$$f'(R)R - 2f(R) + 3\square f' = \kappa T$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}^{(curv)} + T_{\mu\nu}^{(m)}$$

$$T_{\mu\nu}^{(m)} = \frac{\mathcal{X}T_{\mu\nu}}{2f'(R)},$$

$$T_{\mu\nu}^{(curv)} = \frac{1}{f'(R)} \{g_{\mu\nu} [f(R) - Rf'(R)] + f'(R)^{;\rho\sigma} (g_{\mu\rho}g_{\nu\sigma} - g_{\rho\sigma}g_{\mu\nu})\}$$

Exemplo 1: expansão acelerada do universo (“curvatura fraca”)

SEPB, *Phys. Lett. B* 642, 311 (2006), gr-qc/0608072

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - [\nabla_{\mu}\nabla_{\nu} - g_{\mu\nu}\square]f'(R) = \kappa T_{\mu\nu}$$

(matéria “normal”, a aceleração é gerada pela $f(R)$)

Métrica de FRW + $k = 0$ 

$$\rho = -f' R_{tt} - \frac{f}{2} + 3f'' \frac{\dot{a}\dot{R}}{a}$$

$$R_{tt} = 3 \frac{\ddot{a}}{a}$$

$$p = -\frac{f'}{3} (R_{tt} + R) + \frac{f}{2} - f'' \left(\ddot{R} - \frac{2\dot{a}\dot{R}}{a} \right) - f''' \dot{R}^2$$

$$R = -6 \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right)$$

Quantidades cinemáticas:

$$a(t) = a_0 \left\{ 1 + H_0 (t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2 + \frac{1}{3!} j_0 H_0^3 (t - t_0)^3 + \frac{1}{4!} s_0 H_0^4 (t - t_0)^4 + O([t - t_0]^5) \right\}.$$

$$H = \frac{\dot{a}}{a}$$

$$H_0 = 72 \pm 8 \frac{\text{km}}{\text{sMpc}}$$

**Freedman *et al.* ,
ApJ, 553, 47 (2001)**

$$q = -a^{-1} \frac{\ddot{a}}{H^2}$$

$$q_0 = -0.82 \pm 0.14$$

**Rapetti *et al.*,
astro-ph/0605683**

$$j = a^{-1} \frac{\dddot{a}}{H^3}$$

(jerk)

$$j_0 = 2.16 \pm 0.81$$

$$s = \frac{1}{H^4} \frac{\overline{\ddot{a}}}{a}$$

(snap)

$$s_0 = ?$$

$$\rho = -f' R_{tt} - \frac{f}{2} + 3f'' \frac{\dot{a}\dot{R}}{a}$$

$$R_{tt} = 3 \frac{\ddot{a}}{a}$$

$$p = -\frac{f'}{3} (R_{tt} + R) + \frac{f}{2} - f'' \left(\ddot{R} - \frac{2\dot{a}\dot{R}}{a} \right) - f''' \dot{R}^2$$

$$R = -6 \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right)$$

$$\rho_0 \geq 0$$



$$3q_0 H_0^2 f_0' - \frac{f_0}{2} - 18H_0^4 f_0''(j_0 - q_0 - 2) \geq 0$$

$$p_0 = 0$$



$$s_0 = \frac{f_0'}{6H_0^2 f_0''}(q_0 - 2) + 6H_0^2 \frac{f_0'''}{f_0''}(-q_0 + j_0 - 2)^2 - [q_0(q_0 + 6) + 2(1 + j_0)] - \frac{f_0}{12H_0^4 f_0''}$$

$$f_0 \equiv f(R_0)$$

$$R_0 = 6H_0^2(q_0 - 1)$$

Exemplo:

$$f(R) = R + \alpha \ln \frac{R}{\mu},$$

$$\mu < 0.$$

Nojiri e Odintsov (2003)

$$3q_0 H_0^2 f_0' - \frac{f_0}{2} - 18H_0^4 f_0'' (j_0 - q_0 - 2) \geq 0$$



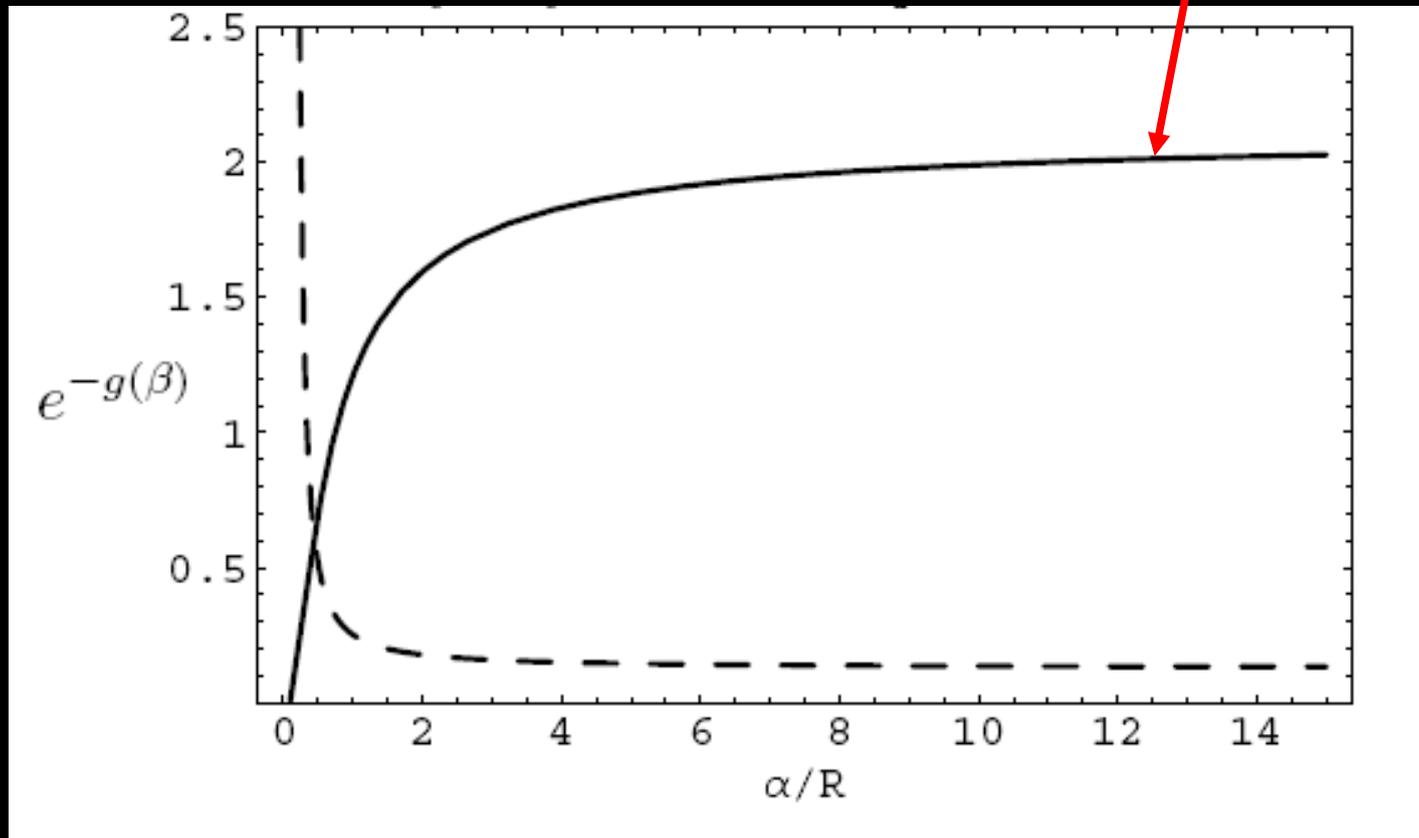
$$0 < \frac{\mu}{R_0} < e^{-g(\beta)}$$

$$g(\beta) = \frac{1}{\beta} \left[-6q_0(1 + \beta) \frac{H_0^2}{R_0} + 1 - 36 \frac{H_0^4}{R_0^2} \beta (j_0 - q_0 - 2) \right]$$

$$\beta = \alpha / R_0$$

$$3q_0 H_0^2 f_0' - \frac{f_0}{2} - 18H_0^4 f_0'' (j_0 - q_0 - 2) \geq 0$$

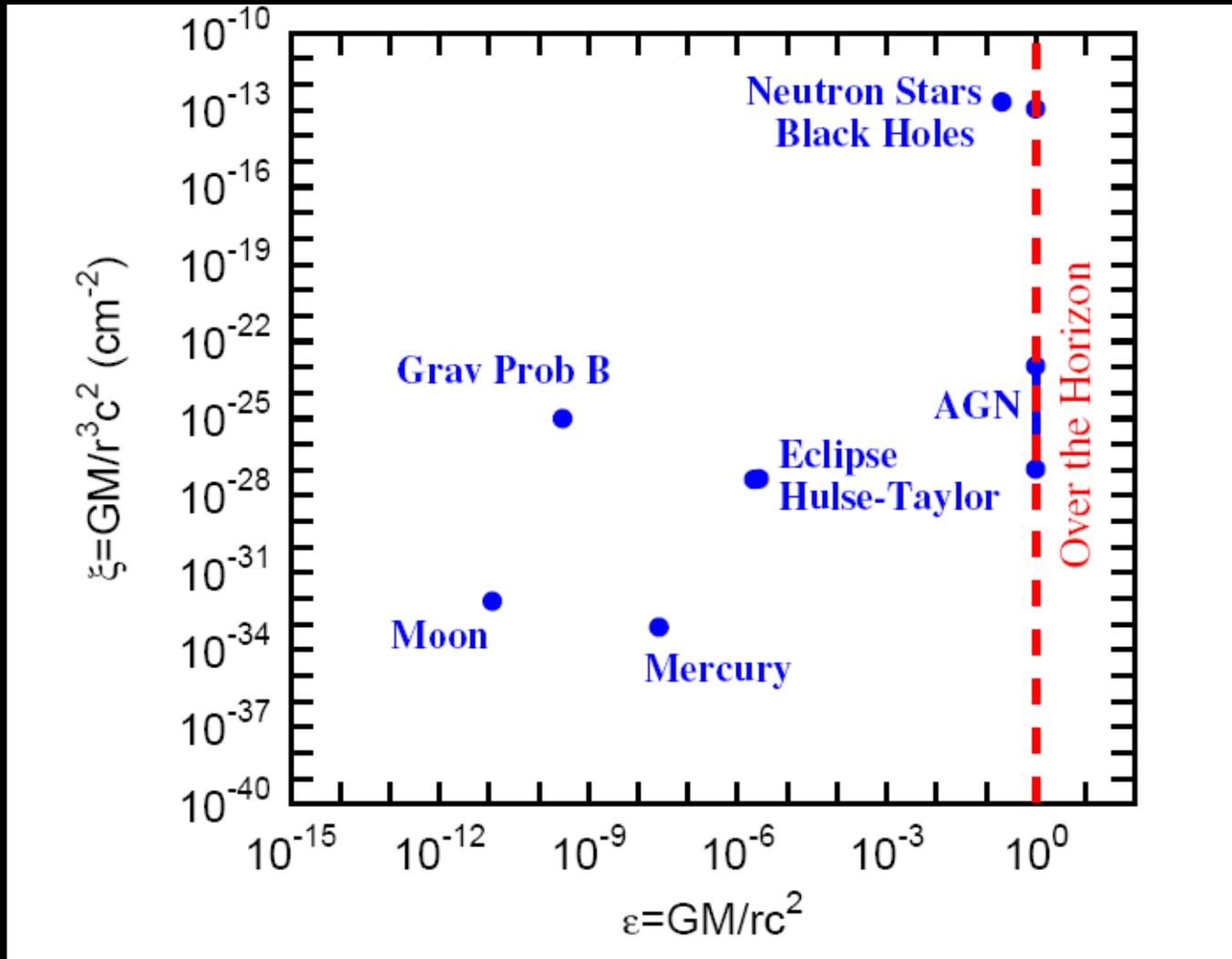
$$|\mu| \lesssim 1.2 \times 10^{-41} \text{ m}^{-2}$$



$$g(\beta) = \frac{1}{\beta} \left[-6q_0(1 + \beta) \frac{H_0^2}{R_0} + 1 - 36 \frac{H_0^4}{R_0^2} \beta (j_0 - q_0 - 2) \right]$$

Exemplo 2: Existência de buracos negros

SEPB e Yves E. Chiafarelli (2009)



$$\xi \equiv \frac{GM}{r^3c^2}$$

Psaltis (2008)

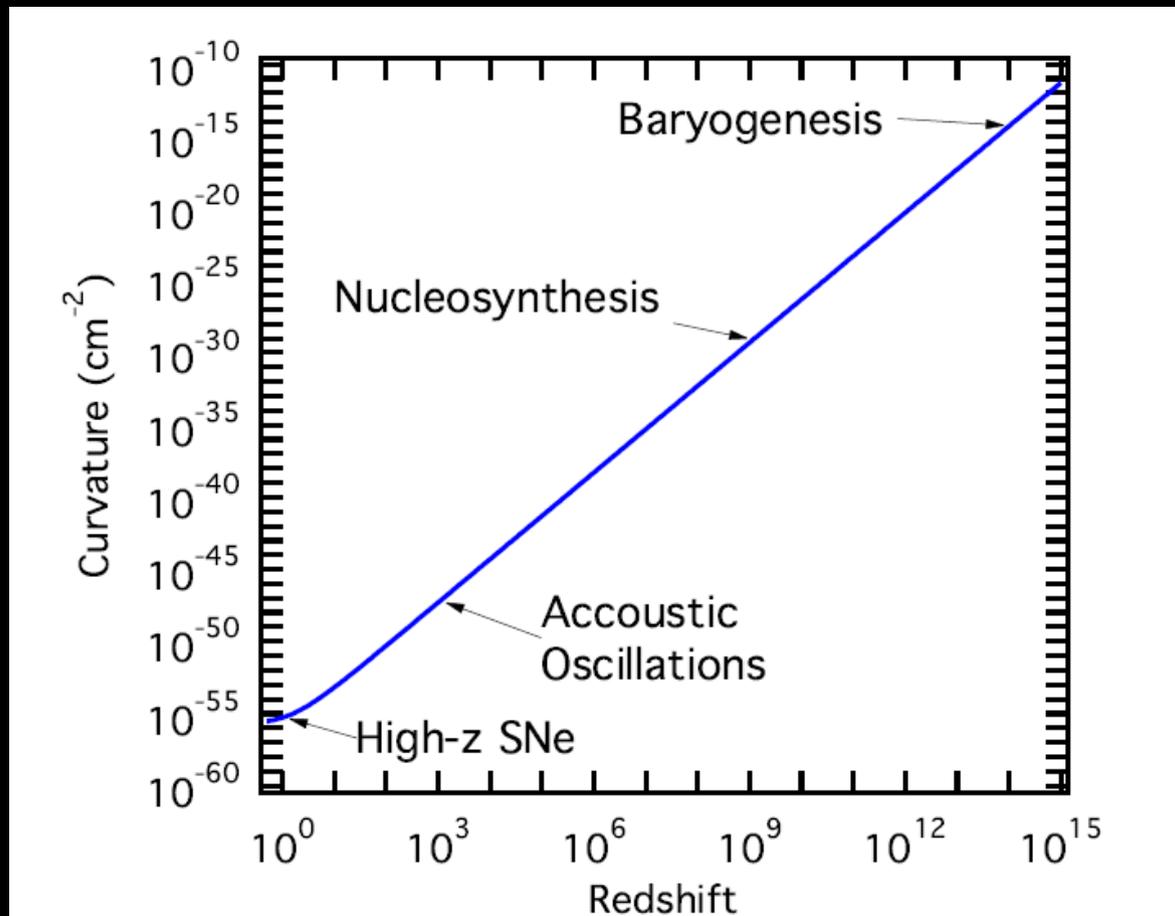


Figure 3: The scalar curvature of our universe, as a function of redshift. The curve corresponds to a flat universe with the best-fit values of the cosmological parameters obtained by the WMAP mission [154]. The arrows point to the curvature and redshift of the universe during various epochs.

Psaltis (2008)

$$ds^2 = -e^{-2\phi(r)} \left(1 - \frac{b(r)}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega^2$$

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - [\nabla_\mu \nabla_\nu - g_{\mu\nu} \square] f'(R) = \kappa T_{\mu\nu}$$

Condições necessárias:

- **Existência de horizonte**

$$ds^2 = -e^{-2\phi(r)} \left(1 - \frac{b(r)}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega^2$$

$$g_{00}(r_0) = 0 \leftrightarrow b(r_0) = r_0$$

- **Forças de maré tipo Schwarzschild no horizonte**

$$\frac{df}{dR} R_{00} + g_{00} A = 0$$

$$\frac{d^2 f}{dR^2} R_{11} + g_{11} A - \frac{d^3 f}{dR^3} (\partial_1 R)^2 - \frac{d^2 f}{dR^2} \nabla_1 (\partial_1 R) = 0$$

$$\frac{df}{dR} R_{22} + g_{22} A = 0$$

$$A \equiv -\frac{f}{2} + \frac{d^3 f}{dR^3} g^{\mu\nu} (\partial_\mu R)(\partial_\nu R) + \frac{d^2 f}{dR^2} \square R$$

$$g_{00}(r_0) = 0 \leftrightarrow b(r_0) = r_0$$

$$b(r) = b_0 + b'_0(r - r_0) + \frac{1}{2}b''_0(r - r_0)^2 + \dots$$

$$\phi(r) = \phi_0 + \phi'_0(r - r_0) + \frac{1}{2}\phi''_0(r - r_0)^2 + \dots$$

$$R_{00} = \frac{1}{2r^2} \left(1 - \frac{b}{r}\right) e^{-2\phi} \left[2r^2 \left(1 - \frac{b}{r}\right) (\phi'' - \phi'^2 + \phi'(4r - b - 3rb' + b''r)) \right]$$

$$\frac{df}{dR} R_{00} + g_{00} A = 0$$

Esta equação é identicamente satisfeita no horizonte (supondo que $f(R)$ é não singular no horizonte)

$$\frac{d^2 f}{dR^2} R_{11} + g_{11} A - \frac{d^3 f}{dR^3} (\partial_1 R)^2 - \frac{d^2 f}{dR^2} \nabla_1 (\partial_1 R) = 0$$

$$R_{11} = -\frac{1}{2r^2 \left(1 - \frac{b}{r}\right)} \left\{ 2r^2 \left(1 - \frac{b}{r}\right) (\phi'' - \phi'^2) + 3\phi'(b - rb') + b''r \right\}$$

$$\frac{df}{dR} R_{22} + g_{22} A = 0$$

$$R_{22} = -r\phi' \left(1 - \frac{b}{r}\right) - b'$$

 **TRÊS EQUAÇÕES**

Avaliando no horizonte:

$$-\frac{1}{2r_0} \frac{df}{dR} \Big|_0 \left[3\phi'_0 r_0 (1 - b'_0) + b''_0 r_0 - \frac{2b'_0}{r_0} \right] + \frac{1}{2r_0^4} \frac{d^2 f}{dR^2} \Big|_0 \left\{ \phi''_0 r_0^3 [-5b'_0 (b'_0 - 1) + 3(b'_0 - 1) + 2b'_0] + \right. \\ \left. \phi'_0 r_0^2 (b'_0 - 1) [-3r_0 b''_0 + 2(b'_0 - 1)] + 2r_0^3 \phi_0'^2 (b'_0 - 1)^2 + r_0^2 (b'_0 - 1) [b_0''' r_0 + b''_0 - 4b'_0] \right\} = 0$$

$$-\frac{1}{2r_0} \frac{df}{dR} \Big|_0 \left[2r_0 (\phi''_0 - \phi_0'^2) - 2\phi'_0 \right] - \frac{1}{r_0^4} \frac{d^3 f}{dR^3} \Big|_0 \left[b''_0 (1 - 3\phi'_0 r_0) + (1 - b'_0) (5r_0 \phi''_0 - 2r_0 \phi_0'^2 - 2\phi'_0) + b''_0 r_0 - \frac{4b'_0}{r_0} \right]^2 + \\ \frac{1}{2r_0^4} \frac{d^2 f}{dR^2} \Big|_0 \left\{ \phi''_0 [-20r_0^3 \phi'_0 (b'_0 - 1) + 16r_0^2 (r_0 b''_0 - 1) - 12r_0^2 b'_0] + 2r_0^2 \phi'_0 (3r_0 b_0''' - 5b''_0) + \right. \\ \left. \phi_0'^2 r_0^2 [-4r_0 b''_0 + 8(b'_0 - 1)] - 2r_0^3 b_0^{iv} + 12b''_0 r_0 - 24b'_0 \right\} = 0$$

$$-b'_0 \frac{df}{dR} \Big|_0 + r_0^2 \left(-\frac{f_0}{2} + \frac{d^2 f}{dR^2} \Big|_0 \square R_0 \right) = 0$$

Caso particular:

$$\phi(r) \equiv 0$$



$$ds^2 = - \left(1 - \frac{b(r)}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega^2$$

$$b'_0 = 0$$

$$z(\infty) = \left(1 - \frac{b(r_E)}{r_E} \right)^{-1/2} - 1$$

$$b(r_E) = b(r_0) + b'(r_0)(r_E - r_0) + \dots$$

$$z(\infty) \approx \left(1 - \frac{r_0}{r_E} \right)^{-1/2} (1 - b'_0)^{-1/2} - 1$$

$$-\frac{1}{2r_0} \frac{df}{dR} \Big|_0 \left[3\phi'_0 r_0 (1 - b'_0) + b''_0 r_0 - \frac{2b'_0}{r_0} \right] + \frac{1}{2r_0^4} \frac{d^2 f}{dR^2} \Big|_0 \left\{ \phi''_0 r_0^3 [-5b'_0 (b'_0 - 1) + 3(b'_0 - 1) + 2b'_0] + \right. \\ \left. \phi'_0 r_0^2 (b'_0 - 1) [-3r_0 b''_0 + 2(b'_0 - 1)] + 2r_0^3 \phi_0'^2 (b'_0 - 1)^2 + r_0^2 (b'_0 - 1) [b_0'''' r_0 + b''_0 - 4b'_0] \right\} = 0$$

$$-\frac{1}{2r_0} \frac{df}{dR} \Big|_0 [2r_0(\phi''_0 - \phi_0'^2) - 2\phi'_0] - \frac{1}{r_0^4} \frac{d^3 f}{dR^3} \Big|_0 \left[b''_0 (1 - 3\phi'_0 r_0) + (1 - b'_0)(5r_0 \phi''_0 - 2r_0 \phi_0'^2 - 2\phi'_0) + b''_0 r_0 - \frac{4b'_0}{r_0} \right]^2 + \\ \frac{1}{2r_0^4} \frac{d^2 f}{dR^2} \Big|_0 \left\{ \phi''_0 [-20r_0^3 \phi'_0 (b'_0 - 1) + 16r_0^2 (r_0 b''_0 - 1) - 12r_0^2 b'_0] + 2r_0^2 \phi'_0 (3r_0 b_0'''' - 5b''_0) + \right. \\ \left. \phi_0'^2 r_0^2 [-4r_0 b''_0 + 8(b'_0 - 1)] - 2r_0^3 b_0'''' + 12b''_0 r_0 - 24b'_0 \right\} = 0$$

$$-b'_0 \frac{df}{dR} \Big|_0 + r_0^2 \left(-\frac{f_0}{2} + \frac{d^2 f}{dR^2} \Big|_0 \square R_0 \right) = 0$$

$$\left. \frac{d^2 f}{dR^2} \right|_0 = - \frac{r_0^2 b_0''}{b_0''' r_0 + b_0''} \left. \frac{df}{dR} \right|_0$$

$$\left. \frac{d^3 f}{dR^3} \right|_0 = \frac{r_0^2 b_0^{iv} - 6b_0''}{(b_0''' r_0 + b_0'')^3} r_0^3 b_0 \left. \frac{df}{dR} \right|_0$$

$$f_0 = \frac{b_0''}{r_0} \left. \frac{df}{dR} \right|_0$$

Forças de maré no horizonte

$$R^{\hat{1}}_{\hat{0}\hat{0}\hat{1}}|_0 = \frac{b''_0 r_0 - 2(b'_0 - 1)}{2r_0^2}$$

$$R^{\hat{2}}_{\hat{0}\hat{0}\hat{2}}|_0 = R^{\hat{3}}_{\hat{0}\hat{0}\hat{3}}|_0 = \frac{b'_0 - 1}{2r_0}$$

Schwarzschild

$$R^{\hat{1}}_{\hat{0}\hat{0}\hat{1}}|_0 > 0$$

$$R^{\hat{2}}_{\hat{0}\hat{0}\hat{2}}|_0 < 0$$

$$b'_0 = 0$$



$$b''_0 > -\frac{2}{r_0}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}^{(curv)} + T_{\mu\nu}^{(m)}$$

$$T_{\mu\nu}^{(m)} = \frac{\mathcal{X}T_{\mu\nu}}{2f'(R)},$$

$$G_{\text{eff}} \equiv \left(\frac{df}{dR} \right)^{-1} G$$



$$f'(R) > 0$$

$$\frac{dG_{\text{eff}}}{dR} = -\frac{\frac{d^2f}{dR^2}}{\left(\frac{df}{dR}\right)^2} G$$



$$f''(R) > 0$$

Dolgov e Kawasaki, 2003

$$g_{00}(r_0) = 0 \leftrightarrow b(r_0) = r_0$$

(horizonte)

$$b''_0 > -\frac{2}{r_0}$$

(forças de maré)

$$b'_0 = 0$$

(redshift)

$$\phi(r) \equiv 0$$

$$\left. \frac{d^2 f}{dR^2} \right|_0 = -\frac{r_0^2 b''_0}{b'''_0 r_0 + b''_0} \left. \frac{df}{dR} \right|_0$$

$$\left. \frac{d^3 f}{dR^3} \right|_0 = \frac{r_0^2 b''_0 - 6b''_0}{(b'''_0 r_0 + b''_0)^3} r_0^3 b_0 \left. \frac{df}{dR} \right|_0$$

(equações de movimento)

$$f_0 = \frac{b''_0}{r_0} \left. \frac{df}{dR} \right|_0$$

$$\left. \frac{df}{dR} \right|_0 > 0$$

(sinal de G_{eff})

(estabilidade)

$$\left. \frac{d^2 f}{dR^2} \right|_0 > 0$$

$$f_0 < 0$$

Condição necessária, dada uma teoria que satisfaça as hipóteses anteriores, para ter um buraco negro da forma

$$ds^2 = - \left(1 - \frac{b(r)}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega^2$$

$$b'_0 = 0$$

Temos ainda que

$$\frac{-2}{r_0} < b''_0 < 0$$

$$b'''_0 > 0$$

Próximos passos:

- **Condições sobre a $f(R)$ para a métrica geral (com b e Φ não nulos)**
- **Soluções com rotação?**
- **Estrelas compactas?**
- **Singularidade (Frolov, 2008) ?**
- **Cálculo das soluções na classe das $f(R)$ “permitidas” e comparação com a observação (“strong lensing”, emissão de ondas gravitacionais, ...)**