

# Appendix of Ferraz-Mello, Michtchenko, Beaugé and Callegari, 2005

## 1 FFT techniques to study chaos

A review of the many techniques used to map chaos is found in Contopoulos (2002; sec. 2.10). We do not review the whole set of techniques here, but just explain the ones used in Ferraz-Mello et al. (2005). Each technique has its own virtues and limitations and often their combination is convenient to get a better understanding of the dynamics of one system. Their common shortcoming is the nonexistence of an exact correlation between their results and the macroscopic instability of the motion. It is our understanding that it is impossible to reduce the qualitative analysis of a planetary system to the blind application of one particular map technique.

The spectral analysis method used in the paper to detect the chaotic domains in a planetary system is based on the well-known features of power spectra (Powell and Percival, 1979). It involves two main steps. The first is the numerical integration of the chosen model, with on-line filtering of the short-periodic terms. The choice of the model requires a careful previous analysis of the system, since computing time in the construction of dynamical maps is large. The application of the digital filtering procedure is an essential step in the construction of dynamical maps. The typical output of a long numerical integration consists of time series of osculating orbital elements that include both short- and long-periodic terms. Since we are interested only in the long-term features of the system behavior, the information about the short-term oscillations is unnecessary. Moreover, these short-period terms generate too large data output and make the identification of the long-term oscillations inefficient. For this reason, the time series of the planets osculating elements should be smoothed by digital filtering, to remove the short-period oscillations (those of the order of the orbital periods). The filtering procedure should be implemented on-line with the numerical integration, as described in detail by Michtchenko and Ferraz-Mello (1995).

The second step of the technique is the spectral analysis of the output of the numerical integrations. The series giving the variation of chosen planetary orbital elements are Fourier-transformed using a standard FFT algorithm. The Fourier transforms of the output allow one to distinguish between regular and chaotic motion.

### 1.1 Regular orbits

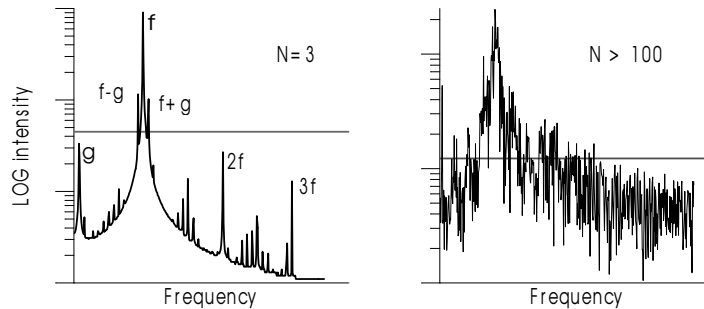
Regular motions are conditionally periodic and any orbital element  $ele(t)$  depends on time as a function:

$$ele(t) = \sum_{\mathbf{k}} A_{\mathbf{k}} \exp(2\pi i \mathbf{k} f t) \quad (1)$$

where  $\mathbf{f}$  is a frequency vector whose components are the fundamental frequencies of motion and  $\mathbf{k} \in \mathbf{Z}^N$ . When the independent frequencies are constant in time, the spectral decomposition of the motion may be obtained from its Fourier transform.

For a smooth function  $ele(t)$ , the amplitudes  $A_{\mathbf{k}}$  decrease rapidly with  $|\mathbf{k}|$ , so that the sum in eqn. (1) is dominated by a few terms. Therefore, the spectrum of regular motion is characterized by a countable (and generally small) number of frequency components. It consists of the lines associated with the independent frequencies, whose number is equal to the number of degrees of freedom of the dynamical system, as well as those corresponding to higher harmonics and to linear combinations of the independent frequencies. The half-width of the each line is of the order of  $\Delta f = 1/T$ , where  $T$  is the time span of the integration.  $T$  defines the transform's resolution: the longer the time  $T$ , the smaller is  $\Delta f$ , and the finer are the details in the Fourier spectrum that can be distinguished. For sufficient large  $T$ , each spectral peak may be approximated by a Dirac  $\delta$ -function.

We illustrate a power spectrum of regular motion in the left panel of fig. 1, which shows the spectrum of the semi-major axis of a regular resonant asteroid orbit obtained with the current initial configuration of the outer Solar System. In this example, the number of significant lines in the power spectrum is equal to 3.



**Fig. 1.** *Left:* Power spectrum of the semi-major axis of a regular orbit. *Right:* Power spectrum of the semi-major axis of a chaotic orbit. In each case,  $N$  is the associated spectral number.

## 1.2 Chaotic orbits

Chaotic motions are no longer conditionally periodic and the fundamental frequencies of the system vary in time. The Fourier transform of the orbital elements is not a sum of Dirac  $\delta$ -functions: the power spectrum of chaotic motion is not discrete, showing broadband components. If the fundamental frequencies variations are enough large and rapid to allow a diffusion over the

chosen time span  $T$  to occur, the power spectrum yields a large quantity of peaks. If the variation of the main frequencies is large and fast enough to be detected on the chosen time span  $T$ , the power spectrum yields a large amount of spectral peaks.

We illustrate a power spectrum of a chaotic motion in the right panel of fig. 1, which shows the spectrum of the semi-major axis of the same asteroid as before when Saturn's semi-major axis is incremented by 0.03 AU to put the Jupiter-Saturn system inside the domain of the 5:2 mean-motions resonance. The number of significant lines in the power spectrum is very large.

## 2 The spectral number

In each power spectrum, we can determine the number of peaks that are above an arbitrarily defined “noise” level. The number thus obtained is called spectral number. In other words, the spectral number  $N$  is the number of significant peaks in the power spectrum of the chosen variable. In general, we consider in this reckoning those peaks with amplitude larger than 5% of the largest peak amplitude. To the case shown in the left panel of fig.1, we associate the spectral number  $N = 3$ , while to the case shown in the right panel, where the number of peaks is so large that an exact reckoning is meaningless, we assign the value of an arbitrary upper limit. (In the cases studied in this paper, we have used  $N = 100$ .)

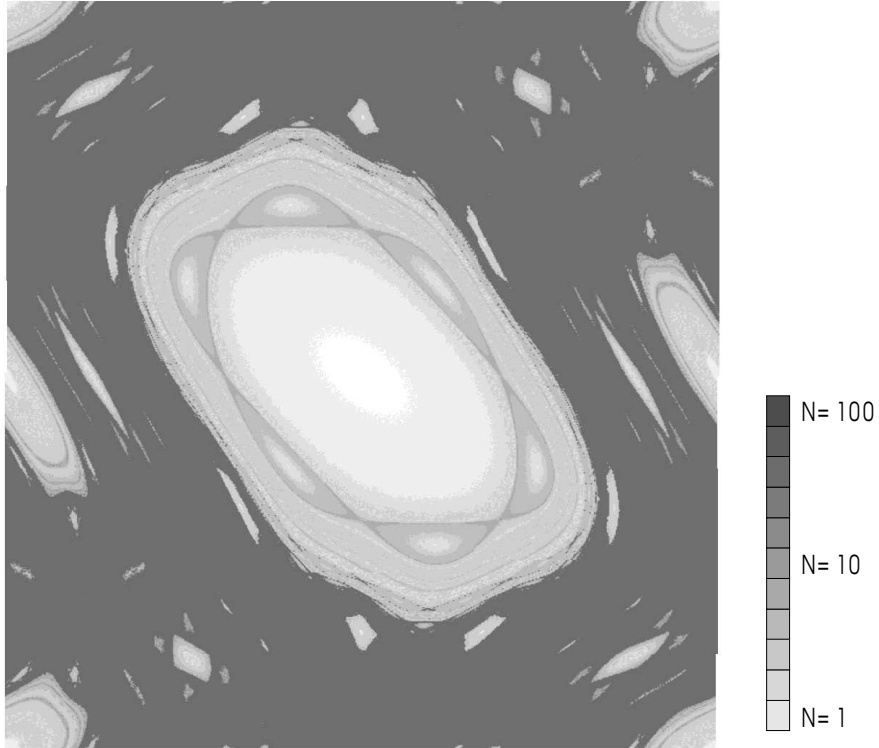
The spectral number  $N$  can be used to qualify the chaoticity of the system in the following way: small values of  $N$  correspond to regular motion, large values of  $N$  indicate the onset of chaos. It should be noted, however, that an orbit classified as regular can appear as chaotic if a larger time span is used in the integrations. Indeed, if the diffusion rate of the main frequencies is below the Fourier Transform resolution (defined by the time span), the spectral analysis methods are unable to detect chaos. The total integration time should be chosen large enough to allow one to distinguish chaos generated by mean motion resonances. Higher order resonances should appear in the dynamical map just by extending the integration time.

## 3 Dynamical Maps

Dynamical maps allowing chaotic domains to be identified are useful tools in the study of the stability of a planetary system. The dynamical maps shown in this paper map the spectral numbers  $N$ . Once  $N$  is determined for all initial conditions on a grid, we plot it on the plane of initial conditions using a gray level scale that varies (logarithmically) from white ( $N = 1$ ) to black ( $N$  maximum). Figure 2 shows the dynamical map of the well-known Taylor-Chirikov standard map

$$\begin{aligned}x_{i+1} &= x_i + \epsilon \sin(x_i + y_i) & (\text{mod } 2\pi) \\y_{i+1} &= x_i + y_i & (\text{mod } 2\pi)\end{aligned}$$

for  $\epsilon = -1.3$ , on a grid of  $401 \times 401$  initial conditions. Since large values of  $N$  indicate the onset of chaos, the gray tones indicate degrees of stochasticity of solutions with initial conditions starting at the map points: lighter regions correspond to regular motion, darker tones indicate chaotic motion (see the discussion in next paragraph). One may appreciate the finesse of details shown by the dynamical map inside and around the main regularity islands of the map.



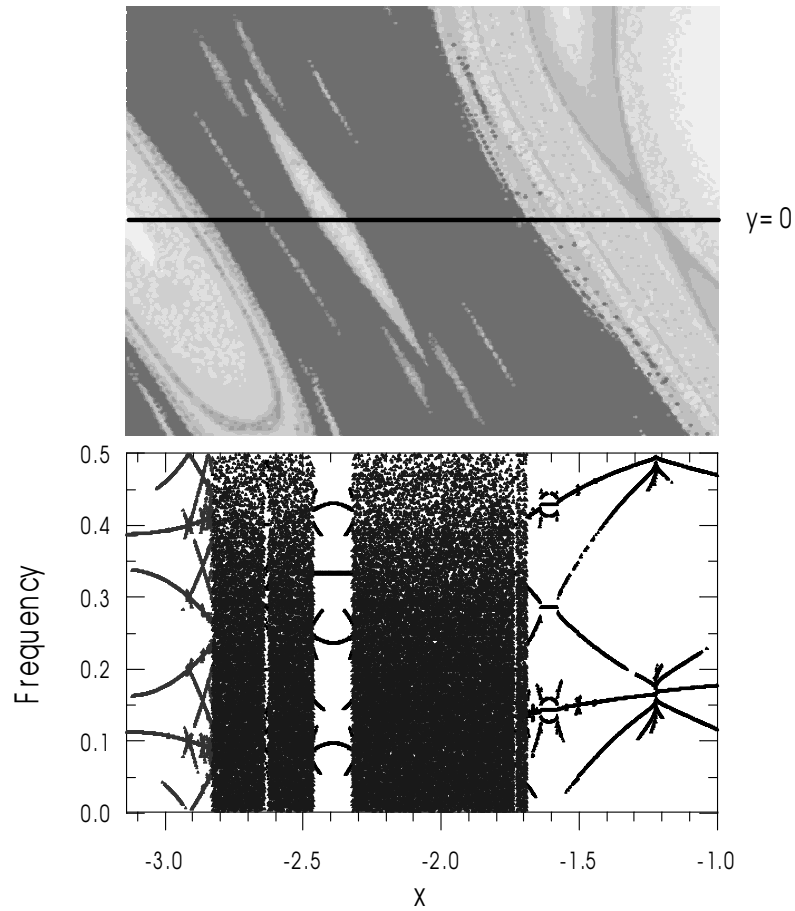
**Fig. 2.** Dynamical Map of the standard map for  $\epsilon = -1.3$  on a grid of  $401 \times 401$  initial conditions. For each initial condition, the degree of chaoticity is given by the spectral number using a uniform logarithmic scale.

## 4 Dynamic power spectrum

Power spectra, as shown in figure 1, are plots of the amplitude of the Fourier Transform against frequency. In order to see how the spectra change when initial conditions vary, we should plot and compare a large number of spectra. The joint representation of the results thus obtained is cumbersome. In some analyses done in this paper, we have adopted a dynamic power spectrum (or Frequency map) obtained by marking the points where the frequency is larger than a chosen limit above noise level in a plane whose axes are one parameter describing a family of solutions (as abscissas) and the frequency of the peaks. For instance, figure 3 shows the dynamic power spectrum of a set of solutions of the Taylor-Chirikov standard map with initial condition  $y = 0$  (the line shown in the map section in the upper part of fig. 3). The family is parameterized by the initial  $x$ . Obviously, one chaotic solution has a huge number of significant amplitude peaks and appear in the dynamic power spectrum as a dark vertical line, and a chaotic region as a dark vertical band. The periodic orbit inside an island appears as one point at the ordinate corresponding to the periodic orbit frequency. Neighbor solutions still show only this frequency but as the initial condition goes away from this value, new frequencies appear: the arcs at several levels seen around  $x = -2.4$ . The selected island is a thin one and the transition to chaos is abrupt. A more continuous picture is seen at the left of  $x = -2.82$  where the number of points increases continuously up to reach the maximum at the border of the chaotic region. Another interesting feature may be seen at  $x = -1.22$  (which corresponds to a saddle point in the standard map). One dark vertical segment is shown in this position and we may see the confluence of many lines showing the increasing number of frequencies whose amplitude becomes larger than the noise level as the saddle point is approached. The remaining parts are repetitions of these basic behaviors.

In dynamic power spectra, the lines have the same behavior found in frequency analysis: frequencies remain almost constant inside resonance islands, have a vertical displacement when crossing a saddle point and become erratic when a chaotic layer is reached (Laskar, 1993).

Dynamic power spectra are not only important complements when studying chaoticity through dynamical maps. In systems with two degrees of freedom, in which the chaoticity may be studied with the help of Poincaré maps (surfaces of section), dynamic power spectra allow us to understand the dynamics of the systems in areas where the maps show intricate features or in which the features are too thin to be visible. They were used in the discussion of the transition from secular dynamics to the 2:1 MMR (Callegari et al. 2004) and in the study of the low-eccentricity dynamics of the 5:2 MMR (Michtchenko and Ferraz-Mello, 2001).



**Fig. 3.** Dynamic power spectrum of the solutions of the standard map. Each feature in the spectrum corresponds to the initial condition with the same abscissa on the line  $y = 0$  (shown by a horizontal line on the map section in top of the figure). The abscissas in the dynamic power spectrum and in the dynamical map section are the same.

## References

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